On Hirschman and log-Sobolev inequalities in μ-deformed Segal–Bargmann analysis

2006 J. Phys. A: Math. Gen. 39 8631

(http://iopscience.iop.org/0305-4470/39/27/006)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 8631–8662 doi:10.1088/0305-4470/39/27/006

On Hirschman and log-Sobolev inequalities in μ-deformed Segal–Bargmann analysis

Claudio de Jesús Pita Ruiz Velasco[1] **and Stephen Bruce Sontz**[2]

[1] Universidad Panamericana, Mexico City, Mexico
[2] Centro de Investigación en Matemáticas, A C (CIMAT), Guanajuato, Mexico

E-mail: cpita@mx.up.mx and sontz@cimat.mx

Received 8 March 2006
Published 21 June 2006
Online at stacks.iop.org/JPhysA/39/8631

Abstract

We consider a μ-deformation of the Segal–Bargmann transform, which is a unitary map from a μ-deformed ground state representation onto a μ-deformed Segal–Bargmann space. We study the μ-deformed Segal–Bargmann transform as an operator between L^p spaces and then we obtain sufficient conditions on the Lebesgue indices for this operator to be bounded. A family of Hirschman inequalities involving the Shannon entropies of a function and of its μ-deformed Segal–Bargmann transform are proved. We also prove a parametrized family of log-Sobolev inequalities, in which a new quantity that we call 'dilation energy' appears. This quantity generalizes the 'energy term' that has appeared in a previous work.

PACS number: 03.65.−w
Mathematics Subject Classification: 81Qxx

1. Introduction

The Segal–Bargmann space $\mathcal{B}^2$ is the Hilbert space of holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ which are square integrable with respect to a Gaussian measure $d\nu_{\text{Gauss}}$. As one thinks of the Hilbert space $L^2(\mathbb{R}, dx)$ as a quantum configuration space, one thinks of $\mathcal{B}^2$ as a quantum phase space, since the spaces $\mathbb{R}$ and $\mathbb{C}$ are the configuration space $\mathbb{R}$ and phase space $T^*\mathbb{R} = \mathbb{R}^2 \cong \mathbb{C}$ for a classical particle with one degree of freedom. In each of the quantum spaces $L^2(\mathbb{R}, dx)$ and $\mathcal{B}^2$ one has unbounded operators a^* (creation) and a (annihilation), which satisfy the relation $[a, a^*] = I$, called the canonical commutation relation (CCR), and both Hilbert spaces carry irreducible representations of the Lie group generated by the exponentiated form of the CCR. The Stone–von Neumann theorem says that in such a case there exists an essentially unique unitary operator $\widetilde{B} : L^2(\mathbb{R}, dx) \to \mathcal{B}^2$ that intertwines the action of the corresponding creation and annihilation operators. This isomorphism $\widetilde{B}$ is the *Bargmann transform*, and *Segal–Bargmann analysis* has to do mainly with the study of the operators related to the transform $\widetilde{B}$ and spaces of holomorphic functions related to $\mathcal{B}^2$. (The beginnings of this theory date back to the works of Segal [Seg1], [Seg2] and Bargmann [Bar].) When the quantum configuration space is replaced by another unitarily equivalent Hilbert space $L^2(\mathbb{R}, \mathrm{d}g)$, called the *ground state representation* (in which $\mathrm{d}g$ is another Gaussian measure), the resulting transform B that maps the ground state representation unitarily onto the Segal–Bargmann space is called the *Segal–Bargmann transform*, and this is the operator that will be of interest for us in this work. In terms of the operators a^* and a one can define the operators P (momentum) and Q (position), which are unbounded self-adjoint operators that satisfy the commutation relation $[P, Q] = -\mathrm{i}I$, which is implied by the CCR. If $H = 2^{-1}(Q^2 + P^2)$ is the Hamiltonian of the harmonic oscillator, one has also that the operators P and Q satisfy the equations of motion $\mathrm{i}[P, H] = Q$ and $\mathrm{i}[Q, H] = -P$. In 1950, Wigner [Wig] proved that the converse implication is false by exhibiting a family of unbounded operators, labelled by a parameter $\mu > -1/2$, that satisfy the equations of motion but do not satisfy the CCR. Rosenblum and Marron described explicitly (in [Ros1], [Ros2] and [Marr]) a μ-quantum configuration space $L^2(\mathbb{R}, |x|^{2\mu}\,\mathrm{d}x)$, a μ-Segal–Bargmann space $\mathcal{B}^2_\mu$, and a μ-Bargmann transform B_μ which is a unitary onto transformation mapping the former Hilbert space to the latter Hilbert space. This theory can be understood as a μ-deformation of standard Segal–Bargmann analysis with the property that if one sets $\mu = 0$ the standard theory is recovered (see [Snt3]).

The Segal–Bargmann transform B shares with the Fourier transform $\mathcal{F}$ the fact of being a unitary operator between L^2 spaces. This is one of the original motivations in [Snt1] for studying B by using $\mathcal{F}$ as a model. For example, the Fourier transform can be studied as an operator acting on L^p spaces, by looking for pairs of Lebesgue indices p and q for which $\mathcal{F} : L^p(\mathbb{R}, \mathrm{d}x) \to L^q(\mathbb{R}, \mathrm{d}x)$ is a bounded operator. The Hausdorff–Young inequality tells us that for $p \in [1, 2]$ and $q = p'$ (the conjugate index of p), the operator $\mathcal{F}$ is bounded and that $\|\mathcal{F}f\|_{L^q(\mathbb{R},\mathrm{d}x)} \leqslant \|f\|_{L^p(\mathbb{R},\mathrm{d}x)}$. In [Snt1] it is proved that for $1 \leqslant q < 2$ and $p > 1 + q/2$, the Segal–Bargmann transform B is a bounded operator from $L^p(\mathbb{R}, \mathrm{d}g)$ to $L^q(\mathbb{C}, \mathrm{d}\nu_{\mathrm{Gauss}})$. By using the Riesz–Thorin interpolation theorem it is also proved that if p and q are as before, then one has the estimate $\|Bf\|_{L^{q(s)}(\mathbb{C},\mathrm{d}\nu_{\mathrm{Gauss}})} \leqslant C^s\|f\|_{L^{p(s)}(\mathbb{R},\mathrm{d}g)}$, where $((p(s))^{-1}, (q(s))^{-1})$ is a point in the segment connecting $(1/2, 1/2)$ and (p^{-1}, q^{-1}). Observe that this result is in fact a family a Hausdorff–Young type inequalities (with B replacing $\mathcal{F}$). In [Hir], Hirschman proved the inequality $S_{L^2(\mathbb{R},\mathrm{d}x)}(f) + S_{L^2(\mathbb{R},\mathrm{d}x)}(\mathcal{F}f) \leqslant 0$, where $S_{L^2(\mathbb{R},\mathrm{d}x)}(\varphi)$ is the entropy (defined in the next section) of the function $\varphi \in L^2(\mathbb{R}, \mathrm{d}x)$. Following [Hir], in [Snt1] the second author proved the 'Hirschman inequality'

$$C_1 S_{L^2(\mathbb{R},\mathrm{d}g)}(f) \leqslant C_2 S_{L^2(\mathbb{C},\mathrm{d}\nu_{\mathrm{Gauss}})}(Bf) + C_3\|f\|^2_{L^2(\mathbb{R},\mathrm{d}g)}. \tag{1.1}$$

The importance of this inequality is that it constrains the values of the entropy of a function and of its Segal–Bargmann transform. That is, even though the operator B does not preserve entropy (also proved in [Snt1]), the inequality above shows that the two entropies cannot have arbitrary values. At this point we mention that from the point of view of the Hilbert space structure, the ground state representation $L^2(\mathbb{R}, \mathrm{d}g)$ is indistinguishable from the Segal–Bargmann space $\mathcal{B}^2$, since B is a Hilbert space isomorphism. In the case of the Fourier transform, the famous Heisenberg uncertainty principle tells us that the variance of a function f and the variance of its Fourier transform $\mathcal{F}f$ are quantities that cannot vary arbitrarily. Thus, the inequality (1.1) can be understood as a kind of uncertainty principle for Segal–Bargmann analysis. The rest of the work in [Snt1] is about replacing the standard Segal–Bargmann space by a similar 'weighted' space. A Hausdorff–Young type family of inequalities is proved by using Stein's interpolation theorem instead of the Riesz–Thorin theorem. Finally, following the same kind of ideas that lead to the Hirschman inequality, the logarithmic-Sobolev inequality

$$C_1 S_{L^2(\mathbb{R},\mathrm{d}g)}(f) \leqslant C_2 S_{L^2(\mathbb{C},\mathrm{d}\nu_{\mathrm{Gauss}})}(Bf) + C_3\langle f, Nf\rangle_{L^2(\mathbb{R},\mathrm{d}g)} + C_4\|f\|^2_{L^2(\mathbb{R},\mathrm{d}g)} \tag{1.2}$$

is shown, where $\langle f, Nf\rangle_{L^2(\mathbb{R},\mathrm{d}g)}$ is the quadratic form associated with the energy (or number) operator $N = a^*a$ acting in the ground state representation $L^2(\mathbb{R}, \mathrm{d}g)$. Some explanations about why (1.2) is called a 'log-Sobolev inequality' are presented in section 6. The motivation of the present work was whether results similar to (1.1) and (1.2) are also valid in the context of the μ-deformed theory of the Segal–Bargmann transform mentioned above. The answers we obtained are presented here.

We now outline the content of the work. In section 2 we give the definitions and some preliminary results that will be used throughout the work. The Banach spaces introduced in that section, which will be involved in the μ-deformed Segal–Bargmann spaces considered in the work (introduced in section 3), are 'weighted' spaces labelled by a parameter $\lambda > 0$. In the case $\mu = 0$ considered in [Snt1], a weight a is introduced, and this parameter is related with λ by $\lambda = 1 + a$. Also, in the case $p = 2$ and $\mu > -1/2$ in [Marr], a weight α is introduced which can be identified with our parameter λ. The case in which $p \geqslant 1$ and $\mu \geqslant 0$, considered in this work, generalizes the case treated in [Snt1] and the case treated in [Marr] as well.

In section 3 we introduce the μ-deformed objects ('*generalized*' objects, in the nomenclature of Rosenblum and Marron) of Segal–Bargmann analysis with which we will work. So we introduce the μ-deformed ground state representation $L^p(\mathbb{R}, \mathrm{d}g_\mu)$, the λ-weighted μ-deformed Segal–Bargmann space $\mathcal{B}^q_{\mu,\lambda}$ and the μ-deformed Segal–Bargmann transform B_μ, for which we are interested in values of p, q and $\lambda > 0$ such that B_μ is a bounded operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $\mathcal{B}^q_{\mu,\lambda}$.

In section 4 we show that if the Lebesgue indices $1 < p \leqslant \infty$, $1 \leqslant q < \infty$ and the weight $\lambda > 1/2$ are such that the inequalities $p > 1 + q/(2\lambda)$ and $1 \leqslant q < 2\lambda$ hold, then the transform B_μ is a bounded operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $\mathcal{B}^q_{\mu,\lambda}$. Observe that the sufficient conditions for this result depend on λ but not on μ. By setting $\mu = 0$ and $\lambda = 1$ we obtain theorem 3.1 of [Snt1]. The importance of the weight λ in the codomain space is shown by noting that for any $1 < p \leqslant \infty$ and $1 \leqslant q < \infty$, the μ-deformed Segal–Bargmann transform B_μ is always a bounded operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $\mathcal{B}^q_{\mu,\lambda}$ provided λ is large enough, namely $\lambda > \max(q/2, q/(2(p-1))$.

From [Ros1], [Ros2] and [Marr] we know that the μ-deformed Segal–Bargmann transform B_μ is a unitary operator from $L^2(\mathbb{R}, \mathrm{d}g_\mu)$ onto $\mathcal{B}^2_{\mu,\lambda}$, provided $\lambda = 1$. In particular we have that for $p = q = 2$ and $\lambda = 1$, the operator B_μ is bounded. We prove in section 5 that the condition $\lambda = 1$ is also necessary for B_μ to be a unitary operator from $L^2(\mathbb{R}, \mathrm{d}g_\mu)$ to $\mathcal{B}^2_{\mu,\lambda}$.

The discussion about Hausdorff–Young type inequalities and Hirschman inequalities is presented in section 5. In that section we work with the μ-deformed Segal–Bargmann transform B_μ as an operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to the unweighted μ-deformed Segal–Bargmann space $\mathcal{B}^q_\mu$. We take the parameters p and q such that the inequalities $p > 1 + q/2$ and $1 \leqslant q < 2$ hold, which imply that B_μ is a bounded operator. In the case $p = q = 2$ the operator B_μ is also bounded since in this case B_μ is unitary. By applying the Riesz–Thorin interpolation theorem, we obtain estimates of the operator norm of B_μ as an operator from $L^{p_s}(\mathbb{R}, \mathrm{d}g_\mu)$ to $\mathcal{B}^{q_s}_\mu$, $s \in [0, 1]$, where $\left(p_s^{-1}, q_s^{-1}\right)$ is a point in the segment connecting $(2^{-1}, 2^{-1})$ with (p^{-1}, q^{-1}). In this way we obtain a Hausdorff–Young type inequality. This inequality has the property that if we set $s = 0$ in it, the inequality becomes an equality, and this fact plays an important role in the idea (called the 'differentiation technique') in the proof of the Hirschman inequality proved in this section. The inequality we obtain is

$$C_1 S_{L^2(\mathbb{R},dg_\mu)}(f) \leqslant C_2 S_{L^2(\mathbb{C}\times Z_2,d\nu_\mu)}(B_\mu f) + C_3\|f\|^2_{L^2(\mathbb{R},dg_\mu)}. \tag{1.3}$$

If we set $\mu = 0$ we recover the inequality (1.1). (The explanation of why the probability measure space $(\mathbb{C}, d\nu_{\text{Gauss}})$ is recovered by setting $\mu = 0$ and $\lambda = 1$ in the measure space $(\mathbb{C} \times Z_2, d\nu_{\mu,\lambda})$ is included in section 2.) Nevertheless we mention that the proof presented in [Snt1] of the inequality (1.1) works for all $f \in L^2(\mathbb{R}, dg)$, while the proof of (1.3) presented here is only valid for functions f in a dense subspace of $L^2(\mathbb{R}, dg_\mu)$.

In section 6 we prove a log-Sobolev inequality following the same steps of the proof of (1.2) in [Snt1]. That is, by using Stein's interpolation theorem we prove first a weighted Hausdorff–Young type inequality, and then by applying the differentiation technique of [Hir] to it, we get the desired log-Sobolev inequality. In the process of proving (1.2) there appears naturally an energy term $\langle Bf, \widetilde{N}Bf\rangle_{\mathcal{B}^2}$, which is the quadratic form associated with the energy operator $\widetilde{N}$ acting in the Segal–Bargmann space $\mathcal{B}^2$ (see [Snt1], pp 2413–14). But the unitarity of the Segal–Bargmann transform gives us that the energy $\langle Bf, \widetilde{N}Bf\rangle_{\mathcal{B}^2}$ is equal to the energy $\langle f, Nf\rangle_{L^2(\mathbb{R},dg)}$ for $f \in L^2(\mathbb{R}, dg)$, where $N = B^{-1}\widetilde{N}B$ is the energy operator acting on the ground-state representation $L^2(\mathbb{R}, dg)$. It is this latter term which appears in (1.2). In the λ-weighted μ-deformed situation we are dealing with there will appear a new mathematical object that generalizes the energy term $\langle Bf, \widetilde{N}Bf\rangle_{\mathcal{B}^2}$ (corresponding to the Segal–Bargmann transform of $f \in L^2(\mathbb{R}, dg)$). We will call it 'dilation energy' and denote it by $E_{\mu,\lambda}(B_\mu f)$ (corresponding to the μ-deformed Segal–Bargmann transform of $f \in L^2(\mathbb{R}, dg_\mu)$). The log-Sobolev inequality we prove in section 6 is

$$C_1 S_{L^2(\mathbb{C}\times Z_2,d\nu_\mu)}(B_\mu f) + C_2 S_{L^2(\mathbb{R},dg_\mu)}(f) \leqslant C_3 E_{\mu,\lambda}(B_\mu f) + C_4\|f\|^2_{L^2(\mathbb{R},dg_\mu)}. \tag{1.4}$$

As is expected, by setting $\mu = 0$ in (1.4) we can recover the inequality (1.2).

Finally, in section 7 we present some conclusions and indicate some questions that we have left unanswered in this work.

The first author has described in [Pi] a formalism which allows this theory to be developed to the context of $\mathbb{R}^n$ and $\mathbb{C}^n$ in place of $\mathbb{R}$ and $\mathbb{C}$. (See also [B-O].) We have not presented this here, since the ideas are the same as in the case $n = 1$ which we consider.

For more background on these topics and our interest in them, consult the introduction of our recent paper [P-S].

2. Preliminaries

In this section we give the definitions and the notation (as well as some preliminary results) that we will use throughout the work. First, we take $\mu \geqslant 0$ and $\lambda > 0$ to be fixed parameters. The (Coxeter) group $\mathbb{Z}_2$ is the multiplicative group $\{-1, 1\}$, and log is the natural logarithm (base e). We use the convention $0 \log 0 = 0$ (which makes the function $\phi : [0, \infty) \to \mathbb{R}$, $\phi(x) = x \log x$ continuous). We also use the convention that C denotes a constant (a quantity that does not depend on the variables of interest in the context), which may change its value every time it appears. We will use when necessary (without further comment) the elementary inequality $(\alpha + \beta)^r \leqslant C_r(\alpha^r + \beta^r)$, valid for all $r > 0$ and $\alpha, \beta \geqslant 0$. For two positive functions f and g such that $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$, we use the notation $f(x) \cong g(x)$ as $x \to a$. For a given $p \in [1, +\infty]$ we will denote by $p' \in [1, +\infty]$ the Lebesgue dual index of p. We denote by $\mathcal{H}(\mathbb{C})$ the space of holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ with the topology of uniform convergence in compact sets.

We begin by defining the μ-deformed factorial function γ_μ and μ-deformed exponential function $\mathbf{e}_\mu$. Let $\mathbb{N}$ denote the set of positive integers.

Definition 2.1. *The μ-deformed factorial function $\gamma_\mu : \mathbb{N} \cup \{0\} \to \mathbb{R}$ is defined by $\gamma_\mu(0) = 1$ and*

$$\gamma_\mu(n) := (n + 2\mu\theta(n))\gamma_\mu(n-1),$$

where $n \in \mathbb{N}$ and $\theta : \mathbb{N} \to \{0, 1\}$ is the characteristic function of the odd positive integers. The μ-deformed exponential function $\mathbf{e}_\mu : \mathbb{C} \to \mathbb{C}$ is defined by the power series

$$\mathbf{e}_\mu(z) := \sum_{n=0}^{\infty} \frac{z^n}{\gamma_\mu(n)}.$$

We note that $\gamma_0(n) = n!$ and so $\mathbf{e}_0(z) = \exp(z)$. It is clear that the power series in the definition of $\mathbf{e}_\mu(z)$ is absolutely convergent for all $z \in \mathbb{C}$. So the μ-deformed exponential $\mathbf{e}_\mu$ is an entire function. Also note that $\gamma_\mu(n) \geqslant n!$ (since we are assuming that $\mu \geqslant 0$), and thus we have the inequality $\mathbf{e}_\mu(x) \leqslant \exp(x)$ for all real non-negative x.

In [Ros1] (lemma 2.3) it is shown that for $\mu > 0$ and $z \in \mathbb{C}$ one has the following integral representation of the μ-deformed exponential function:

$$\mathbf{e}_\mu(z) = \int_{-1}^{1} \exp(tz)\, \mathrm{d}\sigma_\mu(t), \tag{2.1}$$

where $\mathrm{d}\sigma_\mu$ is the probability measure on $[-1, 1]$ given by

$$\mathrm{d}\sigma_\mu(t) := \frac{1}{B\left(\frac{1}{2}, \mu\right)} (1-t)^{\mu-1}(1+t)^{\mu}\, \mathrm{d}t$$

and where B is the beta function (see [Leb], p 13). Note that $B\left(\frac{1}{2}, \mu\right) > 0$ for $\mu > 0$. From this representation one gets easily the fact that $\mathbf{e}_\mu(x) > 0$ for all $x \in \mathbb{R}$.

Lemma 2.1. *For all $\mu \geqslant 0$ and $q \geqslant 1$ the following inequality holds for all $z \in \mathbb{C}$:*

$$|\mathbf{e}_\mu(z)|^q \leqslant \mathbf{e}_\mu(q \operatorname{Re} z). \tag{2.2}$$

Proof. Observe that if $\mu = 0$ the inequality reduces to a trivial equality for all $q \in \mathbb{R}$. If $q = 1$ and $\mu > 0$, one has, by using the integral representation (2.1) of $\mathbf{e}_\mu(z)$, that

$$|\mathbf{e}_\mu(z)| \leqslant \int_{-1}^{1} |\exp(tz)|\, \mathrm{d}\sigma_\mu(t) = \int_{-1}^{1} \exp(t \operatorname{Re} z)\, \mathrm{d}\sigma_\mu(t) = \mathbf{e}_\mu(\operatorname{Re} z),$$

which proves the validity of the inequality for all $\mu > 0$ and $q = 1$. Thus, it remains to prove the inequality in the case $\mu > 0$ and $q > 1$. Again by using the integral representation (2.1) of $\mathbf{e}_\mu(z)$, Hölder's inequality and the fact that $\mathrm{d}\sigma_\mu$ is a probability measure in $[-1, 1]$, we have that

$$\begin{aligned} |\mathbf{e}_\mu(z)| &\leqslant \left(\int_{-1}^{1} |\exp(tz)|^q\, \mathrm{d}\sigma_\mu(t) \right)^{\frac{1}{q}} \left(\int_{-1}^{1} \mathrm{d}\sigma_\mu(t) \right)^{\frac{1}{q'}} \\ &= \left(\int_{-1}^{1} \exp(qt \operatorname{Re} z)\, \mathrm{d}\sigma_\mu(t) \right)^{\frac{1}{q}} \\ &= (\mathbf{e}_\mu(q \operatorname{Re} z))^{\frac{1}{q}}, \end{aligned}$$

which proves the inequality in this case. □

The following definition is due to Angulo and the second author (see [A-S.2]).

Definition 2.2. *Let* $\lambda > 0$. *We define the measure* $\mathrm{d}\nu_{\mu,\lambda}$ *on the space* $\mathbb{C} \times \mathbb{Z}_2$ *by*

$$\mathrm{d}\nu_{\mu,\lambda}(z, 1) := \lambda \frac{2^{\frac{1}{2}-\mu}}{\pi \Gamma\left(\mu + \frac{1}{2}\right)} K_{\mu-\frac{1}{2}}(\lambda|z|^2) \left|\lambda^{\frac{1}{2}} z\right|^{2\mu+1} \mathrm{d}x\, \mathrm{d}y, \tag{2.3}$$

$$\mathrm{d}\nu_{\mu,\lambda}(z, -1) := \lambda \frac{2^{\frac{1}{2}-\mu}}{\pi \Gamma\left(\mu + \frac{1}{2}\right)} K_{\mu+\frac{1}{2}}(\lambda|z|^2) \left|\lambda^{\frac{1}{2}} z\right|^{2\mu+1} \mathrm{d}x\, \mathrm{d}y, \tag{2.4}$$

where Γ *is the Euler gamma function,* K_α *is the Macdonald function of order* α *(both defined in [Leb]) and* $\mathrm{d}x\,\mathrm{d}y$ *is Lebesgue measure on* $\mathbb{C}$.

By convention, in the case $\lambda = 1$ we will omit the parameter λ in the notation of the measure.

The Macdonald function K_α is the modified Bessel function of the third kind (with purely imaginary argument, as described in [Wat], p 78), which is known to be a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$ and is entire with respect to the parameter α. Nevertheless, our interest will be only in the values and behaviour of this function for $x \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$. For $z \in \mathbb{C}$, $|\arg z| < \pi$ and $\alpha \notin \mathbb{Z}$, the Macdonald function can be defined as

$$K_\alpha(z) = \frac{\pi}{2} \frac{I_{-\alpha}(z) - I_\alpha(z)}{\sin(\alpha\pi)}$$

(see [Leb], p 108), where $I_\alpha(z)$ is the modified Bessel function of the first kind. For $\alpha \in \mathbb{Z}$, we define $K_\alpha(z) = \lim_{\beta\to\alpha} K_\beta(z)$. This expression shows that $K_a(z)$ is an even function of the parameter α. In particular, since $I_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sinh z$ and $I_{-\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cosh z$ (see [Leb], p 112), we have that

$$K_{\pm\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \exp(-z), \tag{2.5}$$

which shows that for $\mu = 0$ the measures defined on $\mathbb{C}$ by (2.3) and (2.4) are the same Gaussian measure:

$$\mathrm{d}\nu_{0,\lambda}(z, 1) = \mathrm{d}\nu_{0,\lambda}(z, -1) = \frac{\lambda}{\pi} \exp(-\lambda|z|^2)\, \mathrm{d}x\, \mathrm{d}y.$$

As is noted in [A-S.2], the last expression, when compared with the Gaussian measure

$$\mathrm{d}\nu_{\mathrm{Gauss},\hbar}(z) := \frac{1}{\pi\hbar} \exp\left(-\frac{|z|^2}{\hbar}\right) \mathrm{d}x\, \mathrm{d}y,$$

this being the measure of the Segal–Bargmann space, allows us to identify λ with $\hbar^{-1}$, where $\hbar > 0$ is Planck's constant. (When $\hbar^{-1} = \lambda = 1$ we write this measure simply as $\mathrm{d}\nu_{\mathrm{Gauss}}$.) We consider Planck's constant as a positive parameter. See [Hall], where $\hbar$ is also identified with a 'time' parameter denoted by t.

By using the formula

$$\int_0^\infty K_\alpha(s) s^{\beta-1}\, \mathrm{d}s = 2^{\beta-2} \Gamma\left(\frac{\beta-\alpha}{2}\right) \Gamma\left(\frac{\beta+\alpha}{2}\right),$$

which holds if $\mathrm{Re}\,\beta > |\mathrm{Re}\,\alpha|$ (see [Wat], p 388), we see that

$$\begin{aligned}\int_{\mathbb{C}} \mathrm{d}\nu_{\mu,\lambda}(z, 1) &= \lambda \frac{2^{\frac{1}{2}-\mu}}{\pi\Gamma\left(\mu + \frac{1}{2}\right)} \int_{\mathbb{C}} K_{\mu-\frac{1}{2}}(\lambda|z|^2) \left|\lambda^{\frac{1}{2}} z\right|^{2\mu+1} \mathrm{d}x\, \mathrm{d}y \\ &= \frac{2^{\frac{1}{2}-\mu}}{\Gamma\left(\mu + \frac{1}{2}\right)} \int_0^\infty K_{\mu-\frac{1}{2}}(s) s^{\mu+\frac{1}{2}}\, \mathrm{d}s \\ &= 1,\end{aligned}$$

(where $s = \lambda r^2, r = |z|$), and

$$\int_{\mathbb{C}} \mathrm{d}\nu_{\mu,\lambda}(z,-1) = \lambda \frac{2^{\frac{1}{2}-\mu}}{\pi\Gamma\left(\mu+\frac{1}{2}\right)} \int_{\mathbb{C}} K_{\mu+\frac{1}{2}}(\lambda|z|^2)\left|\lambda^{\frac{1}{2}} z\right|^{2\mu+1} \mathrm{d}x\,\mathrm{d}y$$
$$= \frac{2^{\frac{1}{2}-\mu}}{\Gamma\left(\mu+\frac{1}{2}\right)} \int_0^\infty K_{\mu+\frac{1}{2}}(s)s^{\mu+\frac{1}{2}}\,\mathrm{d}s$$
$$= \frac{\pi^{\frac{1}{2}}\Gamma(\mu+1)}{\Gamma\left(\mu+\frac{1}{2}\right)}.$$

That is, the measures $\mathrm{d}\nu_{\mu,\lambda}(z,1)$ and $\mathrm{d}\nu_{\mu,\lambda}(z,-1)$ on $\mathbb{C}$ are finite, and moreover the former is a probability measure. Another way of seeing this is given in [A-S.2].

The integral representation

$$K_\alpha(z) = \int_0^\infty \exp(-z\cosh u)\cosh(\alpha u)\,\mathrm{d}u \qquad \operatorname{Re} z > 0$$

(see [Leb], p 119) gives us at once two important properties of the Macdonald function. The first is that $K_\alpha(x) > 0$ for all $x \in \mathbb{R}^+$, and the second is that K_α is a monotone decreasing function for $x \in \mathbb{R}^+$.

We will use the following facts about the asymptotic behaviour of the Macdonald function (see [Leb], pp 110, 136):

$$K_\alpha(x) \cong \frac{2^{|\alpha|-1}\Gamma(|\alpha|)}{x^{|\alpha|}} \qquad \text{as} \qquad x \to 0^+ \qquad \text{if} \quad \alpha \neq 0. \tag{2.6}$$

$$K_0(x) \cong \log\frac{2}{x} \qquad \text{as} \qquad x \to 0^+. \tag{2.7}$$

$$K_\alpha(x) \cong \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \exp(-x) \qquad \text{as} \qquad x \to +\infty \quad \text{for all} \quad \alpha \in \mathbb{R}. \tag{2.8}$$

We will be dealing with the complex Banach spaces $L^p(\Omega, \mathrm{d}\nu)$, where $(\Omega, \mathrm{d}\nu)$ is a measure space and $1 \leqslant p \leqslant \infty$. In fact, the measure spaces $(\Omega, \mathrm{d}\nu)$ involved in this work will always be finite. We will denote the norm of a vector $f \in L^p(\Omega, \mathrm{d}\nu)$ by $\|f\|_{L^p(\Omega,\mathrm{d}\nu)}$. If $(\Omega_i, \mathrm{d}\nu_i)$, $i = 1, 2$, are measure spaces and $p, q \geqslant 1$, the norm of an operator defined in some dense subspace D of $L^p(\Omega_1, \mathrm{d}\nu_1)$ with image in $L^q(\Omega_2, \mathrm{d}\nu_2)$ is defined by

$$\|T\|_{p\to q} := \sup\left\{\|Tf\|_{L^q(\Omega_2,\mathrm{d}\nu_2)} : f \in D, \|f\|_{L^p(\Omega_1,\mathrm{d}\nu_1)} = 1\right\}.$$

This is the *operator norm* of T. Although the corresponding measure spaces $(\Omega_i, \mathrm{d}\nu_i)$, $i = 1, 2$, do not appear in the notation $\|T\|_{p\to q}$, these spaces will be clear from context.

The most important operators we will deal with in this work are operators T from some dense domain D of a space $L^p(X, \mathrm{d}\rho)$ into some space $L^q(Y, \mathrm{d}\sigma)$ (where $(X, \mathrm{d}\rho)$ and $(Y, \mathrm{d}\sigma)$ are finite measure spaces), which are integral kernel operators of the form

$$(Tf)(y) = \int_X \widetilde{T}(x, y) f(x)\,\mathrm{d}\rho(x),$$

where $\widetilde{T} : X \times Y \to \mathbb{C}$ is a measurable function, called the *kernel of the operator* T and usually denoted by the same letter T. We define the *Hille–Tamarkin norm* of the kernel T, denoted by $|||T|||_{p,q}$ (unfortunately with the same ambiguity as that of the operator norm), by

$$|||T|||_{p,q} := \|T_p\|_{L^q(Y,\mathrm{d}\sigma)}, \tag{2.9}$$

where $T_p(y) = \|T(\cdot, y)\|_{L^{p'}(X,\mathrm{d}\rho)}$, $y \in Y$. If $1 < p \leqslant \infty$ and $1 \leqslant q < \infty$, we explicitly have

$$|||T|||_{p,q} = \left\{ \int_Y \left(\int_X |T(x,y)|^{p'} \, \mathrm{d}\rho(x) \right)^{\frac{q}{p'}} \mathrm{d}\sigma(y) \right\}^{\frac{1}{q}}.$$

(Note that $|||T|||_{2,2}$ is the Hilbert–Schmidt norm of T.)

Given a pair of Lebesgue indices $(p,q) \in [1,\infty] \times [1,\infty]$, we say that the integral kernel operator T (as described above) is a *Hille–Tamarkin operator* with respect to the pair (p,q) if the Hille–Tamarkin norm (2.9) is finite. It can be proved that the set of Hille–Tamarkin operators with respect to (p,q) is a complex vector space, that (2.9) defines a norm on it, and that this normed space is in fact a Banach space (see theorem 11.5 of [J]).

We will use also the following two results (see [J], theorems 11.5 and 11.6).

Proposition 2.1. $\|T\|_{p\to q} \leqslant |||T|||_{p,q}$.

This proposition tells us that the Hille–Tamarkin operators with respect a given pair (p,q) are bounded from $L^p(X, \mathrm{d}\rho)$ to $L^q(Y, \mathrm{d}\sigma)$.

Proposition 2.2. *If $|||T|||_{p,q} < \infty$ and $1 \leqslant q < \infty$ and $1 < p \leqslant \infty$, then T is a compact operator from $L^p(X, \mathrm{d}\rho)$ to $L^q(Y, \mathrm{d}\sigma)$.*

We will work with the Banach space $L^p(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$, where $p \geqslant 1$.

Let us consider the space

$$\mathfrak{B}_{p,\mu,\lambda} = \{ f : \mathbb{C} \to \mathbb{C} \mid f_e \in L^p(\mathbb{C}, \mathrm{d}\nu_{\mu,\lambda}|_{\mathbb{C}\times\{1\}}) \text{ and } f_o \in L^p(\mathbb{C}, \mathrm{d}\nu_{\mu,\lambda}|_{\mathbb{C}\times\{-1\}})\},$$

where $f = f_e + f_o$ is the decomposition of f in its even and odd parts. Here and subsequently we identify these two restrictions of $\mathrm{d}\nu_{\mu,\lambda}$ as measures on $\mathbb{C}$, using $\mathbb{C} \cong \mathbb{C} \times \{1\} \cong \mathbb{C} \times \{-1\}$. Moreover, we will use without further comment the notation f_e (f_o) for the even part (the odd part, respectively) of a function f.

For $p \geqslant 1$ and $f \in \mathfrak{B}_{p,\mu,\lambda}$, we define

$$\|f\|^p_{\mathfrak{B}_{p,\mu,\lambda}} := \|f_e\|^p_{L^p(\mathbb{C}, \mathrm{d}\nu_{\mu,\lambda}|_{\mathbb{C}\times\{1\}})} + \|f_o\|^p_{L^p(\mathbb{C}, \mathrm{d}\nu_{\mu,\lambda}|_{\mathbb{C}\times\{-1\}})}.$$

The linear map $\Phi : \mathfrak{B}_{p,\mu,\lambda} \to L^p(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$ defined as $(\Phi f)(z,1) = f_e(z)$ and $(\Phi f)(z,-1) = f_o(z)$ is injective and has the property that

$$\|f\|_{\mathfrak{B}_{p,\mu,\lambda}} = \|\Phi f\|_{L^p(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})} \tag{2.10}$$

for all $f \in \mathfrak{B}_{p,\mu,\lambda}$. Therefore $\|\cdot\|_{\mathfrak{B}_{p,\mu,\lambda}}$ is a norm on $\mathfrak{B}_{p,\mu,\lambda}$. It is not hard to show that the range of Φ is a closed subspace of $L^p(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$ for $p \geqslant 1$. (The proof is similar to one found in [Hall].) Therefore $\mathfrak{B}_{p,\mu,\lambda}$ is a Banach space, since we have identified it with a closed subspace of the Banach space $L^p(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$. For a function $f \in \mathfrak{B}_{p,\mu,\lambda}$ we will sometimes write its norm as $\|f\|_{L^p(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})}$, meaning that we are using (2.10) and identifying f with Φf.

We will use the notations $\mathrm{d}\nu_{e,\mu,\lambda}$ and $\mathrm{d}\nu_{o,\mu,\lambda}$ for the restrictions $\mathrm{d}\nu_{\mu,\lambda}|_{\mathbb{C}\times\{1\}}$ and $\mathrm{d}\nu_{\mu,\lambda}|_{\mathbb{C}\times\{-1\}}$, respectively. So for $f \in \mathfrak{B}_{p,\mu,\lambda}$ we have

$$\|f\|^p_{\mathfrak{B}_{p,\mu,\lambda}} = \|f_e\|^p_{L^p(\mathbb{C}, \mathrm{d}\nu_{e,\mu,\lambda})} + \|f_o\|^p_{L^p(\mathbb{C}, \mathrm{d}\nu_{o,\mu,\lambda})} = \|f_e\|^p_{\mathfrak{B}_{p,\mu,\lambda}} + \|f_o\|^p_{\mathfrak{B}_{p,\mu,\lambda}}.$$

Observe that this says that $\mathfrak{B}_{p,\mu,\lambda} = \mathfrak{B}_{e,p,\mu,\lambda} \oplus \mathfrak{B}_{o,p,\mu,\lambda}$, where

$$\mathfrak{B}_{e,p,\mu,\lambda} = \{ f \in \mathfrak{B}_{p,\mu,\lambda} \mid f = f_e \}$$

and

$$\mathfrak{B}_{o,p,\mu,\lambda} = \{ f \in \mathfrak{B}_{p,\mu,\lambda} \mid f = f_o \}$$

are Banach subspaces of $\mathfrak{B}_{p,\mu,\lambda}$.

Let us consider the dilation operator $T_\lambda(f)(z) = f\big(\lambda^{\frac{1}{2}} z\big)$. Let us see that T_λ is an isometry from $\mathfrak{B}_{p,\mu}$ onto $\mathfrak{B}_{p,\mu,\lambda}$. Observe that

$$\int_{\mathbb{C}} |f(z)|^p K_{\mu-\frac{1}{2}}(|z|^2)|z|^{2\mu+1}\, \mathrm{d}x\, \mathrm{d}y = \int_{\mathbb{C}} \big|f\big(\lambda^{\frac{1}{2}} z\big)\big|^p \lambda K_{\mu-\frac{1}{2}}(\lambda|z|^2)\big|\lambda^{\frac{1}{2}} z\big|^{2\mu+1}\, \mathrm{d}x\, \mathrm{d}y$$

by a change of variables. This shows that $T_\lambda f \in \mathfrak{B}_{e,p,\mu,\lambda}$ if and only if $f \in \mathfrak{B}_{e,p,\mu}$, and moreover, that $\|f\|_{\mathfrak{B}_{p,\mu}} = \|T_\lambda f\|_{\mathfrak{B}_{p,\mu,\lambda}}$. Similarly, $T_\lambda f \in \mathfrak{B}_{o,p,\mu,\lambda}$ if and only if $f \in \mathfrak{B}_{o,p,\mu}$, and $\|f\|_{\mathfrak{B}_{p,\mu}} = \|T_\lambda f\|_{\mathfrak{B}_{p,\mu,\lambda}}$. Since clearly $(T_\lambda f)_e = T_\lambda(f_e)$ and $(T_\lambda f)_o = T_\lambda(f_o)$, we have that

$$\begin{aligned} \|f\|^p_{\mathfrak{B}_{p,\mu}} &= \|f_e\|^p_{\mathfrak{B}_{p,\mu}} + \|f_o\|^p_{\mathfrak{B}_{p,\mu}} \\ &= \|T_\lambda(f_e)\|^p_{\mathfrak{B}_{p,\mu,\lambda}} + \|T_\lambda(f_o)\|^p_{\mathfrak{B}_{p,\mu,\lambda}} \\ &= \|T_\lambda f\|^p_{\mathfrak{B}_{p,\mu,\lambda}}, \end{aligned}$$

which proves our claim. In particular, when $p = 2$, the dilation operator T_λ is unitary.

Definition 2.3. *Let* $(\Omega, \mathrm{d}\nu)$ *be a finite measure space, that is,* $0 < \nu(\Omega) < \infty$. *For* $f \in L^2(\Omega, \mathrm{d}\nu)$*, the entropy* $S_{L^2(\Omega,\mathrm{d}\nu)}(f)$ *is defined by*

$$S_{L^2(\Omega,\mathrm{d}\nu)}(f) := \int_\Omega |f(\omega)|^2 \log |f(\omega)|^2\, \mathrm{d}\nu(\omega) - \|f\|^2_{L^2(\Omega,\mathrm{d}\nu)} \log \|f\|^2_{L^2(\Omega,\mathrm{d}\nu)}. \tag{2.11}$$

This definition was introduced by Shannon [Sha] in his Theory of Communication. Note that, since $(\Omega, \mathrm{d}\nu)$ is a finite measure space, the entropy $S_{L^2(\Omega,\mathrm{d}\nu)}(f)$ makes sense for all $f \in L^2(\Omega, \mathrm{d}\nu)$. Moreover, by considering the convex function $\phi : [0, \infty) \to \mathbb{R}$, $\phi(x) = x \log x$, and the probability measure space $(\Omega, \mathrm{d}\nu')$, where $\mathrm{d}\nu' = W^{-1}\, \mathrm{d}\nu$, $W = \nu(\Omega)$, we have by Jensen's inequality (see [L-L], p 38) that

$$\left(\int_\Omega |f(\omega)|^2\, \mathrm{d}\nu(\omega)\right) \log\left(\frac{1}{W}\int_\Omega |f(\omega)|^2\, \mathrm{d}\nu(\omega)\right) \leqslant \int_\Omega |f(\omega)|^2 \log |f(\omega)|^2\, \mathrm{d}\nu(\omega)$$

or

$$(-\log W)\|f\|^2_{L^2(\Omega,\mathrm{d}\nu)} \leqslant S_{L^2(\Omega,\mathrm{d}\nu)}(f),$$

which shows that $S_{L^2(\Omega,\mathrm{d}\nu)}(f) \neq -\infty$, though $S_{L^2(\Omega,\mathrm{d}\nu)}(f) = +\infty$ can happen. Also observe that $S_{L^2(\Omega,\mathrm{d}\nu')}(f) \geqslant 0$, though $S_{L^2(\Omega,\mathrm{d}\nu)}(f)$ can be negative. Finally, note that $S_{L^2(\Omega,\mathrm{d}\nu)}(f)$ is homogeneous of degree 2.

3. The λ-weighted μ-deformed Segal–Bargmann space and its transform

We begin by defining the Segal–Bargmann space of interest for us in this work.

Definition 3.1. *Let* $1 \leqslant q < \infty$. *The* λ*-weighted* μ*-deformed Segal–Bargmann space, denoted by* $\mathcal{B}^q_{\mu,\lambda}$*, is defined as*

$$\mathcal{B}^q_{\mu,\lambda} := \mathcal{H}(\mathbb{C}) \cap \mathfrak{B}_{q,\mu,\lambda}.$$

Although this definition makes sense for $0 < q < \infty$, we will only be interested in the case $1 \leqslant q < \infty$, since in this case the space $\mathcal{B}^q_{\mu,\lambda}$ (the holomorphic subspace of the Banach space $\mathfrak{B}_{q,\mu,\lambda}$) is a Banach space with the norm of $\mathfrak{B}_{q,\mu,\lambda}$.

If we decompose the space $\mathcal{H}(\mathbb{C})$ of holomorphic functions $f : \mathbb{C} \to \mathbb{C}$, as $\mathcal{H}(\mathbb{C}) = \mathcal{H}_e(\mathbb{C}) \oplus \mathcal{H}_o(\mathbb{C})$, where

$$\mathcal{H}_e(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) : f = f_e\} \qquad \text{and} \qquad \mathcal{H}_o(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) : f = f_o\}$$

are the subspaces of the even and odd functions of $\mathcal{H}(\mathbb{C})$, respectively, then by writing $\mathcal{H}(\mathbb{C}) \ni f = f_e + f_o$, the space $\mathcal{B}^q_{\mu,\lambda}$ is just the space of holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ such that the even part f_e (the odd part f_o) of f is q integrable with respect to the measure $\mathrm{d}\nu_{e,\mu,\lambda}$ (with respect to the measure $\mathrm{d}\nu_{o,\mu,\lambda}$, respectively). That is,

$$\mathcal{B}^q_{\mu,\lambda} = \{f \in \mathcal{H}(\mathbb{C}) : f_e \in L^q(\mathbb{C}, \mathrm{d}\nu_{e,\mu,\lambda}) \text{ and } f_o \in L^q(\mathbb{C}, \mathrm{d}\nu_{o,\mu,\lambda})\}.$$

Yet another way to think of $\mathcal{B}^q_{\mu,\lambda}$ is as

$$\mathcal{B}^q_{\mu,\lambda} = \mathcal{B}^q_{e,\mu,\lambda} \oplus \mathcal{B}^q_{o,\mu,\lambda},$$

where

$$\mathcal{B}^q_{e,\mu,\lambda} = \mathcal{H}(\mathbb{C}) \cap \mathfrak{B}_{e,q,\mu,\lambda} = \mathcal{H}_e(\mathbb{C}) \cap \mathfrak{B}_{q,\mu,\lambda}$$

and

$$\mathcal{B}^q_{o,\mu,\lambda} = \mathcal{H}(\mathbb{C}) \cap \mathfrak{B}_{o,q,\mu,\lambda} = \mathcal{H}_o(\mathbb{C}) \cap \mathfrak{B}_{q,\mu,\lambda}$$

are the even and odd subspaces of $\mathcal{B}^q_{\mu,\lambda}$.

In the case $q = 2$, the inner product of the Hilbert space $\mathcal{B}^2_{\mu,\lambda}$ (from which the norm on $\mathcal{B}^2_{\mu,\lambda}$ defined above comes) is

$$\langle f, g\rangle_{\mathcal{B}^2_{\mu,\lambda}} = \langle f_e, g_e\rangle_{L^2(\mathbb{C},\mathrm{d}\nu_{e,\mu,\lambda})} + \langle f_o, g_o\rangle_{L^2(\mathbb{C},\mathrm{d}\nu_{o,\mu,\lambda})}.$$

We then have that the even subspace $\mathcal{B}^2_{e,\mu,\lambda}$ of the space $\mathcal{B}^2_{\mu,\lambda}$ is orthogonal to its odd subspace $\mathcal{B}^q_{o,\mu,\lambda}$, and $\mathcal{B}^2_{\mu,\lambda} = \mathcal{B}^2_{e,\mu,\lambda} \oplus \mathcal{B}^2_{o,\mu,\lambda}$ as Hilbert spaces. When $\mu = 0$ and $\lambda = 1$ we have the Segal–Bargmann space $\mathcal{B}^2 = \mathcal{H}(\mathbb{C}) \cap L^2(\mathbb{C}, \mathrm{d}\nu_{\mathrm{Gauss}})$ that appears in the 'undeformed' theory (see [Hall]).

Observe that $T_\lambda : \mathcal{B}^2_\mu \to \mathcal{B}^2_{\mu,\lambda}$, $(T_\lambda f)(z) = f\left(\lambda^{\frac{1}{2}} z\right)$ is a unitary operator. This comes from the fact that the dilation operator $T_\lambda : \mathfrak{B}_{2,\mu} \to \mathfrak{B}_{2,\mu,\lambda}$ is unitary (as we proved in the previous section), and the fact that $T_\lambda f \in \mathcal{H}(\mathbb{C})$ if and only if $f \in \mathcal{H}(\mathbb{C})$.

The space $\mathcal{B}^2_\mu$ with $\mu > -\frac{1}{2}$ was studied by Rosenblum [Ros2] and by Marron [Marr]. It is known that $\left\{\xi^\mu_n\right\}_{n=0}^\infty$, where $\xi^\mu_n(z) := (\gamma_\mu(n))^{-\frac{1}{2}} z^n$, is an orthonormal basis of $\mathcal{B}^2_\mu$ (see [Marr], p 15, and [A-S.1]). It follows that $\left\{\chi^\mu_n\right\}_{n=0}^\infty$, where $\chi^\mu_n(z) := (\gamma_\mu(n))^{-\frac{1}{2}} \lambda^{\frac{n}{2}} z^n$, is an orthonormal basis of $\mathcal{B}^2_{\mu,\lambda}$, which is obtained by applying the dilation operator $T_\lambda : \mathcal{B}^2_\mu \to \mathcal{B}^2_{\mu,\lambda}$ to the elements of the basis $\left\{\xi^\mu_n\right\}_{n=0}^\infty$.

Rosenblum and Marron considered the *μ-deformed Bargmann transform* $\widetilde{B}_\mu : L^2(\mathbb{R}, |t|^{2\mu}\,\mathrm{d}t) \to \mathcal{B}^2_\mu$ (which they called the *generalized Segal–Bargmann transform*). This can be defined by $\widetilde{B}_\mu\left(\phi^\mu_n\right) = \xi^\mu_n$, where $\left\{\xi^\mu_n\right\}_{n=0}^\infty$ is the orthonormal basis of the μ-deformed Segal–Bargmann space $\mathcal{B}^2_\mu$ mentioned above, and $\left\{\phi^\mu_n\right\}_{n=0}^\infty$ is the orthonormal basis of $L^2(\mathbb{R}, |t|^{2\mu}\,\mathrm{d}t)$ formed by the *μ-deformed Hermite functions* ϕ^μ_n defined by

$$\phi^\mu_n(t) := \left(\frac{\gamma_\mu(n)}{\Gamma\left(\mu + \frac{1}{2}\right)}\right)^{\frac{1}{2}} \frac{1}{2^{\frac{n}{2}} n!} \exp\left(-\frac{t^2}{2}\right) H^\mu_n(t),$$

where $H^\mu_n(t)$ is the *nth μ-deformed Hermite polynomial* defined by the generating function

$$\exp(-z^2)\mathbf{e}_\mu(2tz) = \sum_{n=0}^\infty H^\mu_n(t) \frac{z^n}{n!}.$$

(It is easy to check that $H^\mu_n(t)$ so defined is in fact a polynomial of degree n in t.) Clearly $\widetilde{B}_\mu$ is a unitary map from the μ-deformed quantum configuration space $L^2(\mathbb{R}, |t|^{2\mu}\,\mathrm{d}t)$ onto the μ-deformed quantum phase space $\mathcal{B}^2_\mu$. We mention that the parameter μ in the work of Rosenblum and Marron takes values in $\left(-\frac{1}{2}, +\infty\right)$, and not only in $[0, +\infty)$ as we are considering in this work. As far as we know, the inequality of lemma 2.1 is valid only for non-negative values of μ. This lemma is used in the proof of the main result of the next section (theorem 4.1), and this result in turn plays a fundamental role in the statement and proof of the theorems of sections 5 and 6.

An explicit formula for $\widetilde{B}_\mu$ is (see [Marr], p 16)

$$(\widetilde{B}_\mu f)(z) = \frac{1}{\left(\Gamma\left(\mu+\frac{1}{2}\right)\right)^{\frac{1}{2}}} \exp\left(-\frac{z^2}{2}\right) \int_{\mathbb{R}} f(t)\mathbf{e}_\mu\left(2^{\frac{1}{2}}tz\right) \exp\left(-\frac{t^2}{2}\right) |t|^{2\mu}\, \mathrm{d}t.$$

The point of view we will adopt here (as in [Snt1]) is to replace the μ-deformed quantum configuration space $L^2(\mathbb{R}, |t|^{2\mu}\,\mathrm{d}t)$ by another unitarily equivalent space $L^2(\mathbb{R}, \mathrm{d}g_\mu)$, known as the *$\mu$-deformed ground state representation*, where $\mathrm{d}g_\mu(t) := \left(\phi_0^\mu(t)\right)^2|t|^{2\mu}\,\mathrm{d}t$ and $\phi_0^\mu(t) = \left(\Gamma\left(\mu+\frac{1}{2}\right)\right)^{-\frac{1}{2}} \exp\left(-\frac{t^2}{2}\right)$ is the *ground state* (the first element of the orthonormal basis $\left\{\phi_n^\mu\right\}_{n=0}^\infty$ of $L^2(\mathbb{R}, |t|^{2\mu}\,\mathrm{d}t)$ mentioned above). Note that $\mathrm{d}g_\mu$ is a probability measure that generalizes the Gaussian probability measure $\mathrm{d}g(t) := \pi^{-\frac{1}{2}} \exp(-t^2)\,\mathrm{d}t$ that appears in the case $\mu = 0$ (see [Hall], p 25). Explicitly $\mathrm{d}g_\mu$ looks like

$$\mathrm{d}g_\mu(t) = \left(\Gamma\left(\mu+\tfrac{1}{2}\right)\right)^{-1} \exp(-t^2)|t|^{2\mu}\,\mathrm{d}t. \tag{3.1}$$

Also, it is clear that $G : L^2(\mathbb{R}, |t|^{2\mu}\,\mathrm{d}t) \to L^2(\mathbb{R}, \mathrm{d}g_\mu)$ defined as

$$(Gf)(t) = \left(\Gamma\left(\mu+\frac{1}{2}\right)\right)^{\frac{1}{2}} \exp\left(\frac{t^2}{2}\right) f(t) = \frac{f(t)}{\phi_0^\mu(t)}$$

is a unitary onto map, and then $B_\mu = \widetilde{B}_\mu \circ G^{-1} : L^2(\mathbb{R}, \mathrm{d}g_\mu) \to \mathcal{B}_\mu^2$ is also a unitary map from the μ-deformed ground state representation $L^2(\mathbb{R}, \mathrm{d}g_\mu)$ onto the μ-deformed Segal–Bargmann space $\mathcal{B}_\mu^2$. It is easy to see, from the explicit formula for $\widetilde{B}_\mu$ and (3.1), that an explicit formula for B_μ is

$$(B_\mu f)(z) = \exp\left(-\frac{z^2}{2}\right) \int_{\mathbb{R}} \mathbf{e}_\mu\left(2^{\frac{1}{2}}tz\right) f(t)\,\mathrm{d}g_\mu(t).$$

We will call the transform $B_\mu : L^2(\mathbb{R}, \mathrm{d}g_\mu) \to \mathcal{B}_\mu^2$, defined by the formula above, the *μ-deformed Segal–Bargmann transform*. Observe that if we set $\mu = 0$ this formula becomes

$$(B_0 f)(z) = \int_{\mathbb{R}} \exp\left(-\frac{z^2}{2} + 2^{\frac{1}{2}}tz\right) f(t)\,\mathrm{d}g(t),$$

which is the 'usual' Segal–Bargmann transform studied, for example, in [Hall], where it is shown that is a unitary map from the quantum configuration space $L^2(\mathbb{R}, \mathrm{d}g)$ (the ground-state representation) onto the quantum phase space $\mathcal{B}^2 = \mathcal{H}(\mathbb{C}) \cap L^2(\mathbb{C}, \mathrm{d}\nu_{\mathrm{Gauss}})$ (the Segal–Bargmann space).

For example, let us consider the function $f_n(t) = t^n$ which lies in $L^2(\mathbb{R}, \mathrm{d}g_\mu)$ for any integer $n \geqslant 0$. The μ-deformed Segal–Bargmann transform of f_n is

$$(B_\mu f_n)(z) = \left(\Gamma\left(\mu+\frac{1}{2}\right)\right)^{-1} \exp\left(-\frac{z^2}{2}\right) \int_{\mathbb{R}} \mathbf{e}_\mu\left(2^{\frac{1}{2}}tz\right)t^n \exp(-t^2)|t|^{2\mu}\,\mathrm{d}t.$$

To evaluate this we will use

$$\int_{\mathbb{R}} \mathbf{e}_\mu\,(-\mathrm{i}xt)\, t^n \exp(-t^2)|t|^{2\mu}\,\mathrm{d}t = \frac{(-\mathrm{i})^n\Gamma\left(\mu+\frac{1}{2}\right)\gamma_\mu(n)}{2^n n!} \exp\left(-\frac{x^2}{4}\right) H_n^\mu\left(\frac{x}{2}\right)$$

(see [Ros1], p 378). Then we have that

$$(B_\mu f_n)(z) = \frac{\gamma_\mu(n)}{n!}\left(-\frac{\mathrm{i}}{2}\right)^n H_n^\mu\left(2^{-\frac{1}{2}}\mathrm{i}z\right).$$

For example, if $n = 0$ we have $H_0^\mu(t) = 1$ and then $(B_\mu f_0)(z) = 1$. If $n = 1$ we have $H_1^\mu(t) = \frac{2}{1+2\mu}t$ and then $(B_\mu f_1)(z) = 2^{-\frac{1}{2}}z$. If $n = 2$ we have $H_2^\mu(t) = \frac{4}{1+2\mu}t^2 - 2$ and then $(B_\mu f_2)(z) = \frac{1}{2}z^2 + \frac{1+2\mu}{2}$, and so on. It is clear that B_μ maps polynomials of degree n in $L^2(\mathbb{R}, \mathrm{d}g_\mu)$ to polynomials of degree n in $\mathcal{B}_\mu^2$.

Writing B_μ as an integral kernel operator (and, as usual, writing the kernel also as B_μ) we have that

$$(B_\mu f)(z) = \int_{\mathbb{R}} B_\mu(z, t) f(t)\, \mathrm{d}g_\mu(t), \tag{3.2}$$

where the kernel $B_\mu : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is

$$B_\mu(z, t) = \exp\left(-\frac{z^2}{2}\right) \mathbf{e}_\mu\left(2^{\frac{1}{2}}tz\right).$$

For each $z = x + \mathrm{i}y \in \mathbb{C}$ fixed, let us consider the function $t \mapsto B_\mu(z, t)$. If $1 < p \leqslant \infty$ we have that

$$\begin{aligned}
&\int_{\mathbb{R}} |B_\mu(z, t)|^{p'}\, \mathrm{d}g_\mu(t) \\
&\quad= \left(\Gamma\left(\mu + \frac{1}{2}\right)\right)^{-1} \left|\exp\left(-\frac{z^2}{2}\right)\right|^{p'} \int_{\mathbb{R}} \left|\mathbf{e}_\mu\left(2^{\frac{1}{2}}tz\right)\right|^{p'} \exp(-t^2)|t|^{2\mu}\, \mathrm{d}t \\
&\quad\leqslant \left(\Gamma\left(\mu + \frac{1}{2}\right)\right)^{-1} \exp\left(-p'\frac{x^2 - y^2}{2}\right) \int_{\mathbb{R}} \mathbf{e}_\mu\left(2^{\frac{1}{2}}p'tx\right) \exp(-t^2)|t|^{2\mu}\, \mathrm{d}t \\
&\quad= \exp\left(\frac{p'}{2}(p' - 1)x^2 + \frac{p'}{2}y^2\right) < \infty,
\end{aligned}$$

where we used the inequality (2.2) and the equality

$$\int_{\mathbb{R}} \mathbf{e}_\mu\left(\pm 2^{\frac{1}{2}}p'xt\right) \exp(-t^2)|t|^{2\mu}\, \mathrm{d}t = \Gamma\left(\mu + \frac{1}{2}\right) \exp\left(\frac{p'^2x^2}{2}\right), \tag{3.3}$$

which comes from the formula

$$\int_{\mathbb{R}} \mathbf{e}_\mu(-\mathrm{i}\tilde{x}t)\mathbf{e}_\mu(\mathrm{i}\tilde{y}t) \exp(-\eta t^2)|t|^{2\mu}\, \mathrm{d}t = \frac{\Gamma\left(\mu + \frac{1}{2}\right)}{\eta^{\mu+\frac{1}{2}}} \exp\left(-\frac{\tilde{x}^2 + \tilde{y}^2}{4\eta}\right) \mathbf{e}_\mu\left(\frac{\tilde{x}\tilde{y}}{2\eta}\right)$$

(see [Ros1], p 379) with $\tilde{x} = \pm\mathrm{i}2^{\frac{1}{2}}p'x$, $\tilde{y} = 0$ and $\eta = 1$. This shows that the function $t \mapsto B_\mu(z, t)$ belongs to the space $L^{p'}(\mathbb{R}, \mathrm{d}g_\mu)$.

Observe that if $f \in L^p(\mathbb{R}, \mathrm{d}g_\mu)$, $1 < p \leqslant \infty$, we have by Hölder's inequality that

$$\int_{\mathbb{R}} |B_\mu(z, t) f(t)|\, \mathrm{d}g_\mu(t) \leqslant \left(\int_{\mathbb{R}} |B_\mu(z, t)|^{p'}\, \mathrm{d}g_\mu(t)\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}} |f(t)|^p\, \mathrm{d}g_\mu(t)\right)^{\frac{1}{p}} < \infty.$$

That is, $(B_\mu f)(z)$ defined in (3.2) makes sense for any $f \in L^p(\mathbb{R}, \mathrm{d}g_\mu)$, $1 < p \leqslant \infty$ and any $z \in \mathbb{C}$. Observe that Morera's theorem tells us that $B_\mu f : \mathbb{C} \to \mathbb{C}$ is holomorphic. The goal of the next section will be to identify values of $p \in (1, +\infty]$, $q \in [1, +\infty)$ and $\lambda > 0$ such that B_μ is a bounded operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $\mathcal{B}_{\mu,\lambda}^q$. For example, we know that when $p = q = 2$, $\lambda = 1$ (and $\mu \geqslant 0$, a situation included in the work of Rosenblum and Marron), the operator B_μ is bounded, since in this case B_μ is an isometry. But as we will see in the next section, there are 'lots' of pairs of Lebesgue indices $(p, q) \in (1, +\infty] \times [1, +\infty)$

(or equivalently $(p^{-1}, q^{-1}) \in [0, 1) \times (0, 1]$, with the standard conventions $0^{-1} = +\infty$ and $+\infty^{-1} = 0$), and values of the parameter $\lambda > 0$, for which B_μ is a bounded operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $\mathcal{B}^q_{\mu,\lambda}$.

What we will do in the next section is to obtain sufficient conditions on the Lebesgue indices p and q, and on the weight $\lambda > 0$ for B_μ to be a Hille–Tamarkin operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^q(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$. The rest of this section is devoted to making some observations which will simplify the work of the proof of theorem 4.1.

Observe that, for any $f \in L^p(\mathbb{R}, \mathrm{d}g_\mu)$ given, we can write the decomposition of the function $B_\mu f$ in its even and odd parts as

$$(B_\mu f)(z) = (B_\mu f)_e(z) + (B_\mu f)_o(z) = \int_{\mathbb{R}} B_{e,\mu}(z,t) f(t)\, \mathrm{d}g_\mu(t) + \int_{\mathbb{R}} B_{o,\mu}(z,t) f(t)\, \mathrm{d}g_\mu(t),$$

where

$$B_{e,\mu}(z,t) = \frac{1}{2}\exp\left(-\frac{z^2}{2}\right)\left(\mathbf{e}_\mu\left(2^{\frac{1}{2}}zt\right) + \mathbf{e}_\mu\left(-2^{\frac{1}{2}}zt\right)\right)$$

and

$$B_{o,\mu}(z,t) = \frac{1}{2}\exp\left(-\frac{z^2}{2}\right)\left(\mathbf{e}_\mu\left(2^{\frac{1}{2}}zt\right) - \mathbf{e}_\mu\left(-2^{\frac{1}{2}}zt\right)\right)$$

are the even and odd parts of $z \mapsto B_\mu(z,t)$, respectively. Thus, we can consider operators $B_{e,\mu}$ and $B_{o,\mu}$ defined for all $f \in L^p(\mathbb{R}, \mathrm{d}g_\mu)$, as $B_{e,\mu} f = (B_\mu f)_e$ and $B_{o,\mu} f = (B_\mu f)_o$, that is

$$(B_{e,\mu} f)(z) = \int_{\mathbb{R}} B_{e,\mu}(z,t) f(t)\, \mathrm{d}g_\mu(t),$$

$$(B_{o,\mu} f)(z) = \int_{\mathbb{R}} B_{o,\mu}(z,t) f(t)\, \mathrm{d}g_\mu(t).$$

So $B_{e,\mu}$ and $B_{o,\mu}$ are integral kernel operators whose kernels are the even and odd parts of the kernel of the integral kernel operator B_μ. Suppose that there exist $p \in (1, +\infty]$, $q \in [1, +\infty)$ and $\lambda > 0$ such that B_μ is a bounded operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^q(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$. (We will see in the next section that such p, q, λ do exist.) Then we have that $B_{e,\mu}$ is a bounded operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^q(\mathbb{C}, \mathrm{d}\nu_{e,\mu,\lambda})$ and $B_{o,\mu}$ is a bounded operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^q(\mathbb{C}, \mathrm{d}\nu_{o,\mu,\lambda})$. Conversely, if there exist $p \in (1, +\infty]$, $q \in [1, +\infty)$ and $\lambda > 0$ such that $B_{e,\mu}$ and $B_{o,\mu}$ are bounded operators from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^q(\mathbb{C}, \mathrm{d}\nu_{e,\mu,\lambda})$ and to $L^q(\mathbb{C}, \mathrm{d}\nu_{o,\mu,\lambda})$, respectively, then B_μ is a bounded operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^q(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$.

Finally, let us note that since $B_\mu = B_{e,\mu} + B_{o,\mu}$, we have that $|||B_\mu|||_{p,q} \leqslant |||B_{e,\mu}|||_{p,q} + |||B_{o,\mu}|||_{p,q}$, so if $B_{e,\mu}$ and $B_{o,\mu}$ are Hille–Tamarkin operators with respect to p and q, then B_μ is a Hille–Tamarkin operator with respect to p and q.

4. L^p mapping properties of B_μ

The main result in this section is the following.

Theorem 4.1. *Let $1 < p \leqslant \infty$, $1 \leqslant q < \infty$ and $\lambda > \frac{1}{2}$. A sufficient condition for B_μ to be a Hille–Tamarkin operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^q(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$ is that p, q and λ satisfy the inequalities*

$$p > 1 + \frac{q}{2\lambda} \qquad \text{and} \qquad 1 \leqslant q < 2\lambda. \tag{4.1}$$

(Note that these conditions do not depend on μ.)

Remark. In the case $\mu = 0$ these conditions are also necessary for the operator to be Hille–Tamarkin. See [Snt1]. We conjecture that this is also true in this more general context.

Proof. As we mentioned in the last section, it is sufficient to prove that the conditions (4.1) imply that $B_{e,\mu}$ and $B_{o,\mu}$ are Hille–Tamarkin operators with respect to p and q. We begin by considering $B_{e,\mu}$. We have that

$$|B_{e,\mu}(z,t)| = \left|\frac{1}{2}\exp\left(-\frac{z^2}{2}\right)\left(\mathbf{e}_\mu(2^{\frac{1}{2}}zt) + \mathbf{e}_\mu(-2^{\frac{1}{2}}zt)\right)\right|$$
$$\leqslant \exp\left(-\frac{x^2-y^2}{2}\right)\left(\left|\mathbf{e}_\mu(2^{\frac{1}{2}}zt)\right| + \left|\mathbf{e}_\mu(-2^{\frac{1}{2}}zt)\right|\right),$$

where $z = x + \mathrm{i}y \in \mathbb{C}$, $x, y \in \mathbb{R}$. Note that this inequality is also valid for the kernel $B_{o,\mu}(z,t)$. By using (2.2) and (3.3) we have that

$$\left(\int_{\mathbb{R}} |B_{e,\mu}(z,t)|^{p'}\,\mathrm{d}g_\mu(t)\right)^{\frac{1}{p'}}$$
$$\leqslant \left\{\int_{\mathbb{R}}\left(\exp\left(-\frac{x^2-y^2}{2}\right)\left(\left|\mathbf{e}_\mu(2^{\frac{1}{2}}zt)\right| + \left|\mathbf{e}_\mu(-2^{\frac{1}{2}}zt)\right|\right)\right)^{p'}\mathrm{d}g_\mu(t)\right\}^{\frac{1}{p'}}$$
$$\leqslant C\exp\left(-\frac{x^2-y^2}{2}\right)\left(\int_{\mathbb{R}}\left(\left|\mathbf{e}_\mu(2^{\frac{1}{2}}zt)\right|^{p'} + \left|\mathbf{e}_\mu(-2^{\frac{1}{2}}zt)\right|^{p'}\right)\mathrm{d}g_\mu(t)\right)^{\frac{1}{p'}}$$
$$\leqslant C\exp\left(-\frac{x^2-y^2}{2}\right)\left(\int_{\mathbb{R}}\left(\mathbf{e}_\mu(p'2^{\frac{1}{2}}xt) + \mathbf{e}_\mu(-p'2^{\frac{1}{2}}xt)\right)\exp(-t^2)|t|^{2\mu}\,\mathrm{d}t\right)^{\frac{1}{p'}}$$
$$= C\exp\left(-\frac{x^2-y^2}{2} + \frac{p'x^2}{2}\right).$$

Thus we obtain

$$|||B_{e,\mu}|||_{p,q} = \left(\int_{\mathbb{C}}\left(\int_{\mathbb{R}}|B_{e,\mu}(z,t)|^{p'}\,\mathrm{d}g_\mu(t)\right)^{\frac{q}{p'}}\mathrm{d}\nu_{e,\mu,\lambda}(z)\right)^{\frac{1}{q}}$$
$$\leqslant C\left(\int_{\mathbb{C}}\exp\left(-q\frac{x^2-y^2}{2} + \frac{qp'x^2}{2}\right)K_{\mu-\frac{1}{2}}(\lambda|z|^2)|z|^{2\mu+1}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{q}}.$$

The last integral is finite if and only if for all $M > 0$ we have that

$$\int_{|z|>M}\exp\left(-q\frac{x^2-y^2}{2} + \frac{qp'x^2}{2}\right)K_{\mu-\frac{1}{2}}(\lambda|z|^2)|z|^{2\mu+1}\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

But for large enough $M > 0$ we can use the asymptotic behaviour given in (2.8) of $K_{\mu-\frac{1}{2}}(\lambda|z|^2)$ as $|z| \to \infty$ (which does not depend on the order of the Macdonald function) to conclude that the last expression is equivalent to

$$\int_{|z|>M}\exp\left(\left(-\frac{q}{2} + \frac{qp'}{2} - \lambda\right)x^2 + \left(\frac{q}{2} - \lambda\right)y^2\right)(x^2+y^2)^\mu\,\mathrm{d}x\,\mathrm{d}y < \infty,$$

which is equivalent to the conditions

$$-\frac{q}{2} + \frac{qp'}{2} - \lambda < 0 \qquad \text{and} \qquad \frac{q}{2} - \lambda < 0,$$

which are the conditions in the hypotheses of the theorem. We have proved that these conditions guarantee that $B_{e,\mu}$ is a Hille–Tamarkin operator. But, as we mentioned before, the same estimates obtained for $|||B_{e,\mu}|||_{p,q}$ work for $|||B_{o,\mu}|||_{p,q}$, since the Macdonald function $K_{\mu+\frac{1}{2}}(\lambda|z|^2)$ also has the same asymptotics as $|z| \to \infty$. So the same conditions guarantee that $B_{o,\mu}$ is a Hille–Tamarkin operator. So finally we conclude that the conditions on p, q and λ in the theorem imply that B_μ is a Hille–Tamarkin operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^q(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$, as desired. □

We have proved that for p, q and λ as in (4.1), the Hille–Tamarkin norm of B_μ is finite, and then proposition 2.1 allows us to conclude the boundedness of $B_\mu : L^p(\mathbb{R}, \mathrm{d}g_\mu) \to L^q(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$. Observe that even though we do not have the case $p = 2, q = 2$ and $\lambda = 1$ included in (4.1), we do have the boundedness of B_μ since for these values of p, q and λ the operator B_μ is in fact unitary. In other words, the conditions imposed by the inequalities (4.1) are sufficient to conclude the boundedness of B_μ, but those conditions are not necessary. On the other hand, proposition 2.2 (together with theorem 4.1) tells us that the inequalities (4.1) are also sufficient to conclude that B_μ is a compact operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^q(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$. The natural question is if in the case $p = 2, q = 2$ and $\lambda = 1$ the operator B_μ is compact. The answer is no. In fact, we know that the Segal–Bargmann transform $B : L^2(\mathbb{R}, \mathrm{d}g) \to \mathcal{B}^2$ *is not a compact operator*, since in this case B is a unitary map onto the *infinite-dimensional* space $\mathcal{B}^2$. Thus, even though we have that $\|B_\mu\|_{2\to 2} = 1$, we have that $|||B_\mu|||_{2,2} = \infty$ (again by proposition 2.2).

The case $\mu = 0$ and $\lambda = 1$ of theorem 4.1 is contained in theorem 3.1 of [Snt1]. So we have that if $1 < p \leqslant \infty, 1 \leqslant q < \infty$ are such that the inequalities $p > 1 + \frac{q}{2}$ and $1 \leqslant q < 2$ hold, then the Segal–Bargmann transform $B : L^2(\mathbb{R}, \mathrm{d}g) \to \mathcal{B}^2$ is bounded. But in this case we have more: if either $p < 1 + \frac{q}{2}$ or $q > 2$ holds, the Segal–Bargmann transform B is unbounded (see corollary 7.2 in [Snt1]).

The pair $(p^{-1}, q^{-1}) \in [0, 1) \times (0, 1]$ is called *admissible* if $\|B_\mu\|_{p\to q} < \infty$.

The inequalities (4.1) can be written as

$$q^{-1} > \frac{1}{2\lambda}\frac{p^{-1}}{1 - p^{-1}} \qquad \text{and} \qquad \frac{1}{2\lambda} < q^{-1} \leqslant 1. \tag{4.2}$$

In the plane with points (p^{-1}, q^{-1}), the curve

$$q^{-1} = \frac{1}{2\lambda}\frac{p^{-1}}{1 - p^{-1}}$$

is a hyperbola with vertical asymptote $p^{-1} = 1$ and horizontal asymptote $q^{-1} = -\frac{1}{2\lambda}$. This hyperbola passes through the origin and intersects the horizontal line $q^{-1} = 1$ in $\left(\frac{2\lambda}{2\lambda+1}, 1\right)$. Then, if R is the region determined by the inequalities (4.2), we have $R \cap ([0, 1) \times (0, 1]) \neq \varnothing$, which shows the existence of a non-empty region of admissible pairs (p^{-1}, q^{-1}) for which the μ-deformed Segal–Bargmann transform is a bounded operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^q(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$. Note that the condition $\lambda > \frac{1}{2}$ guarantees the existence of $q^{-1} \in (0, 1]$ satisfying the inequality $\frac{1}{2\lambda} < q^{-1} \leqslant 1$ of (4.2).

We observe that the fact that (p^{-1}, q^{-1}) is an admissible pair depends on the value of λ. For example, if $\lambda = \frac{2}{3}$, the pair $\left(\frac{1}{4}, \frac{4}{5}\right)$ is admissible, since for these values of λ, p, and q the inequalities (4.2) hold. Also, the pair $\left(\frac{1}{4}, \frac{2}{5}\right)$ is admissible for $\lambda = 2$, but it is not, for example, for $\lambda = 1$. (Certainly one easily checks that for $p^{-1} = \frac{1}{4}, q^{-1} = \frac{2}{5}$ and $\lambda = 1$ the inequalities (4.2) do not hold. But as we have seen before this does not imply that the pair $\left(\frac{1}{4}, \frac{2}{5}\right)$ is not admissible for $\lambda = 1$. The conclusion comes from corollary 7.2 in [Snt1] mentioned above, since in this case we have $q > 2$.) So we have that if the weight λ of

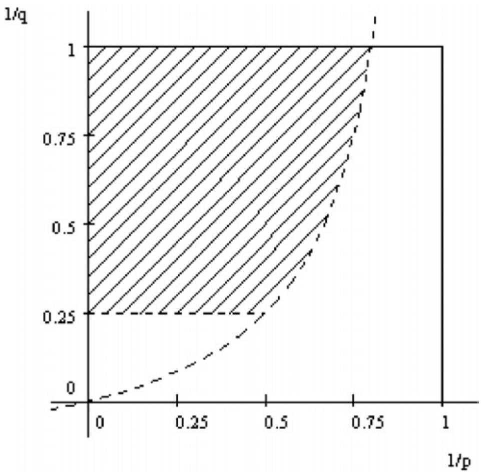

Figure 1. Region where (4.2) holds for $\lambda = 2 : \frac{1}{4} < q^{-1} \leqslant 1, q^{-1} > \frac{p^{-1}}{4(1-p^{-1})}$.

the codomain space $L^q(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$ is fixed and $\lambda > \frac{1}{2}$, then we always can find pairs (p^{-1}, q^{-1}) (those that satisfy (4.2)) for which the μ-deformed Segal–Bargmann transform B_μ is a bounded operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^q(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$. Moreover, observe that if we have a fixed pair (p^{-1}, q^{-1}) satisfying the inequalities (4.2) for a given $\lambda_1 > \frac{1}{2}$, then these inequalities are also satisfied for any $\lambda \geqslant \lambda_1$.

But there is another point of view of the situation described above: *any pair* $(p^{-1}, q^{-1}) \in [0, 1) \times (0, 1]$ *can be admissible, by taking an adequate value of* λ. In fact, observe that if we take

$$\lambda > \max\left(\frac{1}{2q^{-1}}, \frac{p^{-1}}{2q^{-1}(1-p^{-1})}\right), \tag{4.3}$$

then the inequalities (4.2) are satisfied for any $(p^{-1}, q^{-1}) \in [0, 1) \times (0, 1]$. That is, for any pair $(p, q) \in (1, +\infty] \times [1, +\infty)$, the μ-deformed Segal–Bargmann transform $B_\mu : L^p(\mathbb{R}, \mathrm{d}g_\mu) \to L^q(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$, where λ is taken as in (4.3), is a bounded operator.

Figures 1–3 show the regions of pairs (p^{-1}, q^{-1}) where (4.2) holds in the cases $\lambda = 2, \lambda = 1$ and $\lambda = \frac{2}{3}$, respectively. So these regions are contained in the regions of admissible pairs (p^{-1}, q^{-1}).

5. Hirschman inequalities

We know that $B_\mu : L^2(\mathbb{R}, \mathrm{d}g_\mu) \to \mathcal{B}^2_{\mu,\lambda}$ is a unitary operator when $\lambda = 1$. So the condition $\lambda = 1$ is sufficient for B_μ being unitary. The following result tells us that this condition is also necessary.

Proposition 5.1. *Suppose that the operator B_μ from $L^2(\mathbb{R}, \mathrm{d}g_\mu)$ to $\mathcal{B}^2_{\mu,\lambda}$ is unitary. Then $\lambda = 1$.*

Proof. Let f be a state of $L^2(\mathbb{R}, \mathrm{d}g_\mu)$ (i.e., $\|f\|_{L^2(\mathbb{R},\mathrm{d}g_\mu)} = 1$). By taking the orthonormal basis $\{\xi_n^\mu\}_{n=0}^\infty$ of $\mathcal{B}^2_\mu$ (see section 3), we can write $B_\mu f \in \mathcal{B}^2_\mu$ as $B_\mu f = \sum_{n=0}^\infty a_n \xi_n^\mu$, where the coefficients $a_n \in \mathbb{C}$ satisfy $\sum_{n=0}^\infty |a_n|^2 = 1$ (since in this case B_μ is unitary). We can take f such that $a_k \neq 0$ for some $k \in \mathbb{N}$. Suppose (in order to get a contradiction) that $\lambda > 1$. By

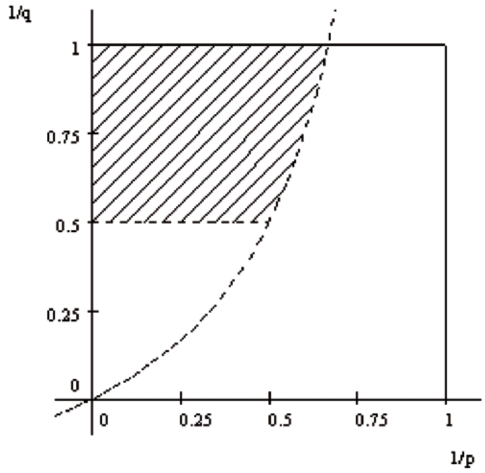

Figure 2. Region where (4.2) holds for $\lambda = 1 : \frac{1}{2} < q^{-1} \leqslant 1, q^{-1} > \frac{p^{-1}}{2(1-p^{-1})}$.

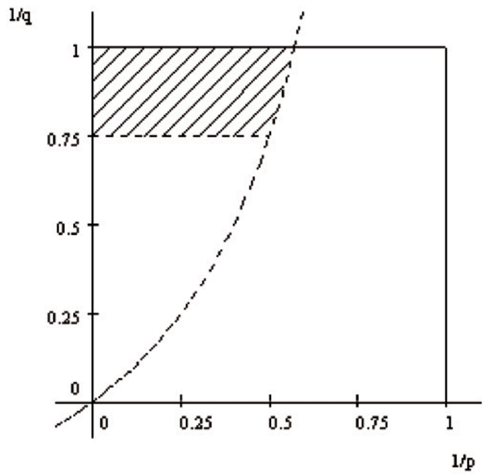

Figure 3. Region where (4.2) holds for $\lambda = \frac{2}{3} : \frac{3}{4} < q^{-1} \leqslant 1, q^{-1} > \frac{3p^{-1}}{4(1-p^{-1})}$.

hypothesis we have that $B_\mu f \in \mathcal{B}^2_{\mu,\lambda}$, so we can write $B_\mu f$ in terms of the basis $\left\{\chi^\mu_n\right\}_{n=0}^\infty$ of $\mathcal{B}^2_{\mu,\lambda}$ (where $\chi^\mu_n = \lambda^{\frac{n}{2}}\xi^\mu_n$) as $B_\mu f = \sum_{n=0}^\infty \lambda^{-\frac{n}{2}} a_n \chi^\mu_n$. Since we are assuming that the operator B_μ from $L^2(\mathbb{R}, \mathrm{d}g_\mu)$ to $\mathcal{B}^2_{\mu,\lambda}$ is unitary, we have that $1 = \sum_{n=0}^\infty \left|\lambda^{-\frac{n}{2}} a_n\right|^2 = \sum_{n=0}^\infty \lambda^{-n} |a_n|^2$, and since $\lambda > 1$ and $a_k \neq 0$ for some $k \in \mathbb{N}$, we have that $1 = \sum_{n=0}^\infty \lambda^{-n} |a_n|^2 < \sum_{n=0}^\infty |a_n|^2 = 1$, a contradiction. A similar contradiction occurs in the case $0 < \lambda < 1$. Thus we conclude that $\lambda = 1$, as desired. □

In the same spirit as the Hausdorff–Young inequality (HYI, for short), which states the boundedness of the Fourier transform $\mathcal{F} : L^p(\mathbb{R}, \mathrm{d}x) \to L^{p'}(\mathbb{R}, \mathrm{d}x)$ for $p \in [1, 2]$ (see [R-S], p 328), as well as other related theorems (see [We], pp 168–9), which also concern boundedness properties of some operators between L^p spaces, we are going to establish an inequality involving the operator norm of B_μ for a range of values of p and q. This result will play a central role in the demonstration of the main result of this section. The most important tool used in the proof of this inequality is the Riesz–Thorin interpolation theorem, which is also used in the demonstrations of the HYI and the other related theorems mentioned above.

Theorem 5.1 (Hausdorff–Young type inequality)**.** *Take* $1 \leqslant q < 2$, $p > 1 + \frac{q}{2}$ *and* p_s *and* q_s *defined by*

$$p_s = (sp^{-1} + (1-s)2^{-1})^{-1}$$

and

$$q_s = (sq^{-1} + (1-s)2^{-1})^{-1}$$

for $s \in [0, 1]$. *Then we have*

$$1 \leqslant \|B_\mu\|_{p_s \to q_s} \leqslant \|B_\mu\|^s_{p \to q}.$$

Proof. Observe that the pairs $(2^{-1}, 2^{-1})$ and (p^{-1}, q^{-1}) are admissible for B_μ and that $\|B_\mu\|_{2\to 2} = 1$ since B_μ is unitary. The Riesz–Thorin interpolation theorem (see [B-S], p 196) says that for any $s \in [0, 1]$, the operator B_μ from $L^{p_s}(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^{q_s}(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_\mu)$ is bounded and that $\|B_\mu\|_{p_s \to q_s} \leqslant \|B_\mu\|^s_{p\to q} \|B_\mu\|^{1-s}_{2\to 2} = \|B_\mu\|^s_{p\to q}$. Moreover, since $B_\mu 1 = 1$, we have also the inequality $\|B_\mu\|_{p_s\to q_s} \geqslant 1$, which completes the proof of the theorem. □

In the main result of this section, which we will present and prove shortly, we will face the problem of differentiating functions of the form $\varphi(s) = \|f\|_{L^{T(s)}(\Omega,\mathrm{d}\nu)}$ at $s = 0$, where $(\Omega, \mathrm{d}\nu)$ is a finite measure space, $T : [0, 1] \to \mathbb{R}$ is the function

$$T(s) = \frac{2\vartheta}{(2-\vartheta)s + \vartheta}, \tag{5.1}$$

$\vartheta \geqslant 1$ is a parameter and f is a non-zero function in the space $L^p(\Omega, \mathrm{d}\nu)$ for $p > 2$. More precisely, we will need to calculate the right-hand derivative $\varphi'(0^+)$, and of course before that, to guarantee its existence.

If we naively calculate $\varphi'(0^+)$, interchanging when necessary the differentiation with integration and applying the rules from elementary calculus, we get

$$\varphi'(0^+) = \left(\frac{1}{2} - \frac{1}{\vartheta}\right) \|f\|^{-1}_{L^2(\Omega,\mathrm{d}\nu)} S_{L^2(\Omega,\mathrm{d}\nu)}(f), \tag{5.2}$$

where $S_{L^2(\Omega,\mathrm{d}\nu)}(f)$ is the entropy of f, defined in (2.11).

Note that by the very definition of $\varphi'(0^+)$ a *necessary condition* for the existence of this derivative is that $\varphi(s)$ be finite in some interval of the form $[0, \varepsilon)$. That is, we need that the function f *belong* to $L^{T(s)}(\Omega, \mathrm{d}\nu)$ for $0 \leqslant s < \varepsilon$. Let us write this necessary condition as (NC). We can guarantee NC, if for example we require that $f \in L^{2+\zeta}(\Omega, \mathrm{d}\nu)$ where $\zeta > 0$, since in this case we have $T(0) = 2 < 2 + \zeta$ which implies that there exists $\varepsilon > 0$ such that $T(s) < 2 + \zeta$ for $0 \leqslant s < \varepsilon$ which in turn implies (using Hölder's inequality) that $\|f\|_{L^{T(s)}(\Omega,\mathrm{d}\nu)} \leqslant C\|f\|_{L^{2+\zeta}(\Omega,\mathrm{d}\nu)} < \infty$ for $0 \leqslant s < \varepsilon$. That is, the condition $f \in L^{2+\zeta}(\Omega, \mathrm{d}\nu)$ where $\zeta > 0$ is a *sufficient condition* (denoted (SC)) for NC. (We mention that SC is not necessary for NC, since if $1 \leqslant \vartheta < 2$ we have that $f \in L^2(\Omega, \mathrm{d}\nu)$ is enough to imply NC as one can easily check.) Surprisingly, the condition SC is also a *sufficient condition* for the existence of $\varphi'(0^+)$, and in such a case formula (5.2) obtained by formal derivation is the *right* formula for this derivative. This is what the following lemma says; it is lemma 1.1 of [G] with some minor changes.

Lemma 5.1. *Let* $(\Omega, \mathrm{d}\nu)$ *be a finite measure space. Suppose* $\varepsilon > 0$, $1 < p_0 < \infty$, *and* $p > p_0$. *Let* $T(s)$ *be a real continuously differentiable function on* $[0, \varepsilon)$ *such that* $T(0) = p_0$, *and let*

$F(s)$ be a continuously differentiable function on $[0, \varepsilon)$ into $L^p(\Omega, \nu)$ with $F(0) = f \neq 0$. Then $\|F(s)\|_{L^{T(s)}(\Omega,\mathrm{d}\nu)}$ is differentiable from the right at $s = 0$ and its derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0^+} \|F(s)\|_{L^{T(s)}(\Omega,\mathrm{d}\nu)} = N^{1-p_0}\left(p_0^{-1}T'\left(0^+\right)\left(\int_\Omega |f|^{p_0}\log|f|\,\mathrm{d}\nu - N^{p_0}\log N\right) + \mathrm{Re}\langle F'(0^+), f_{p_0}\rangle\right), \tag{5.3}$$

where $N = \|f\|_{L^{p_0}(\Omega,\mathrm{d}\nu)}$ and $f_{p_0} = (\mathrm{sgn}\, f)\,|f|^{p_0-1}$.

We emphasize that under these hypotheses, the derivative (5.3) is a finite real number.

The sign of $z \in \mathbb{C}$, denoted by $\mathrm{sgn}\, z$, is defined as $\mathrm{sgn}\, z = z/|z|$ if $z \neq 0$, and $\mathrm{sgn}\, z = 0$ if $z = 0$. In the case we are dealing with, namely $\varphi(s) = \|f\|_{L^{T(s)}(\Omega,\mathrm{d}\nu)}$, we have $p_0 = 2$, $p = 2 + \zeta$ with $\zeta > 0$, $T(s)$ given by (5.1) (so that $T'(0) = -\frac{2}{\vartheta}(2 - \vartheta)$), and F is constant (equal to f for all s, so that $F'(0) = 0$). Thus, if we denote the norm $\|f\|_{L^2(\Omega,\mathrm{d}\nu)}$ by N, formula (5.3) is in our case

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0^+} \|f\|_{L^{T(s)}(\Omega,\mathrm{d}\nu)} &= N^{-1}2^{-1}\left(\frac{-2\,(2-\vartheta)}{\vartheta}\right)\left(\int_\Omega |f|^2\log|f|\,\mathrm{d}\nu - N^2\log N\right) \\
&= \left(\frac{1}{2} - \frac{1}{\vartheta}\right) N^{-1}\left(\int_\Omega |f|^2\log|f|^2\,\mathrm{d}\nu - N^2\log N^2\right) \\
&= \left(\frac{1}{2} - \frac{1}{\vartheta}\right) N^{-1} S_{L^2(\Omega,\mathrm{d}\nu)}(f),
\end{aligned}$$

which is formula (5.2) for $\varphi'(0^+)$.

Roughly speaking, an *uncertainty principle* is an inequality involving the variance of a function f and the variance of its Fourier transform $\mathcal{F}f$. (See [Fol], p 27, for a more general statement of an uncertainty principle.) For example, the Heisenberg uncertainty principle states that for any $f \in L^2(\mathbb{R}, \mathrm{d}x)$ such that $\|f\|_{L^2(\mathbb{R},\mathrm{d}x)} = 1$ one has

$$\left[\left(\int_\mathbb{R} (x-\Lambda)^2|f(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\right]\left[\left(\int_\mathbb{R} (x-\widehat{\Lambda})^2\,|(\mathcal{F}f)(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\right] \geqslant (4\pi)^{-1},$$

where the factors on the left-hand side are the variances of f and of $\mathcal{F}f$ and

$$\Lambda = \int_\mathbb{R} x|f(x)|^2\,\mathrm{d}x \qquad \text{and} \qquad \widehat{\Lambda} = \int_\mathbb{R} x|(\mathcal{F}f)(x)|^2\,\mathrm{d}x$$

(assumed to be finite) are the expected values of f and $\mathcal{F}f$, respectively. What this inequality tells us is that the variances of f and $\mathcal{F}f$ cannot be simultaneously arbitrarily small. Of course, this has to do with the well-known physical version of the Heisenberg uncertainty principle about the impossibility of determining simultaneously position and momentum of a quantum particle.

In his paper [Hir], Hirschman obtained an inequality involving not the variances of f and $\mathcal{F}f$, but their entropies. Specifically, he showed that for $f \in L^2(\mathbb{R}, \mathrm{d}x)$ such that $\|f\|_{L^2(\mathbb{R},\mathrm{d}x)} = 1$ one has

$$S_{L^2(\mathbb{R},\mathrm{d}x)}(f) + S_{L^2(\mathbb{R},\mathrm{d}x)}(\mathcal{F}f) \leqslant 0,$$

whenever the left-hand side has meaning. Note that $(\mathbb{R}, \mathrm{d}x)$ is not a finite measure space, so that one or both of the terms on the left-hand side can be meaningless. In fact, Hirschman conjectured a sharper upper bound, namely $\log 2 - 1$, for the right hand side of the previous inequality. However, Beckner in [Be] proved this conjecture. The idea behind Hirschman's method for proving the inequality above is to view each side of the HYI $\|\mathcal{F}f\|_{p'} \leqslant \|f\|_p$ which is valid for $p \in [1, 2]$, as a function of p for fixed $f \in L^2(\mathbb{R}, \mathrm{d}x)$. It turns out that these functions are smooth, and then it makes sense to take the derivative at $p = 2^-$ on both sides of the inequality. The point is that, when $p = 2$, the HYI is in fact an equality, by Plancherel's theorem, and so the derivative $\frac{\mathrm{d}}{\mathrm{d}p}\big|_{p=2^-}$ acts as an order-reversing operator giving in this way a new inequality, 'the differentiated HYI at $p = 2^-$'. It turns out that this yields Hirschman's result. All these ideas were applied in the context of Segal–Bargmann analysis by the second author ([Snt1]) in the case $\mu = 0$. Following the same kind of ideas, we now establish the main result of this section.

Theorem 5.2 (Hirschman inequality)**.** *Suppose that p and q satisfy*

$$1 \leqslant q < 2 \qquad \textit{and} \qquad p > 1 + \frac{q}{2}.$$

Let $f \in L^{2+\zeta}(\mathbb{R}, \mathrm{d}g_\mu)$ *with* $\zeta > 0$ *be such that* $B_\mu f \in L^{2+\xi}(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_\mu)$ *for some* $\xi > 0$*. Then the Hirschman entropy inequality*

$$(p^{-1} - 2^{-1}) S_{L^2(\mathbb{R},\mathrm{d}g_\mu)}(f) \leqslant (q^{-1} - 2^{-1}) S_{L^2(\mathbb{C}\times\mathbb{Z}_2,\mathrm{d}\nu_\mu)}(B_\mu f) + (\log \|B_\mu\|_{p\to q}) \|f\|^2_{L^2(\mathbb{R},\mathrm{d}g_\mu)} \tag{5.4}$$

holds.

Remark. We comment that the set of functions for which the hypotheses of theorem 5.2 hold is a dense subspace of $L^2(\mathbb{R}, \mathrm{d}g_\mu)$. This is shown in the Remark after theorem 6.3.

Proof. We first note that if $f = 0$ the inequality to prove is trivial, both sides of it being equal to zero. So we take an arbitrary f satisfying the hypotheses with $f \neq 0$. Observe that the coefficient of the norm term in (5.4) is non-negative, since $\|B_\mu\|_{p\to q} \geqslant 1$. So the term itself is non-negative. Nevertheless, the remaining two terms (the entropy terms) can be positive, negative or zero. In fact, even though $S_{L^2(\mathbb{R},\mathrm{d}g_\mu)}(f) \geqslant 0$ (since $(\mathbb{R}, \mathrm{d}g_\mu)$ is a probability measure space), the hypotheses allow the coefficient $(p^{-1} - 2^{-1})$ to be positive, negative or zero. Also, the hypotheses give us that $(q^{-1} - 2^{-1}) > 0$, but the entropy $S_{L^2(\mathbb{C}\times\mathbb{Z}_2,\mathrm{d}\nu_\mu)}(B_\mu f)$ can be positive, negative or zero. (Recall that $(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_\mu)$ is a measure space with weight strictly greater than 1.)

The idea of the proof consists in considering the Hausdorff–Young type inequality $\|B_\mu\|_{p_s\to q_s} \leqslant \|B_\mu\|^s_{p\to q}$ we proved above (theorem 5.1), where $p_s = (sp^{-1} + (1-s)2^{-1})^{-1}$ and $q_s = (sq^{-1} + (1-s)2^{-1})^{-1}$, with $s \in [0, 1]$. Observe that these formulae for p_s and q_s are of the form $T(s) = \frac{2\vartheta}{(2-\vartheta)s+\vartheta}$, with $\vartheta = p$ and $\vartheta = q$, respectively, as in the discussion previous to the theorem. That is, we begin by considering the inequality $\|B_\mu f\|_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2,\mathrm{d}\nu_\mu)} \leqslant A^s \|f\|_{L^{p_s}(\mathbb{R},\mathrm{d}g_\mu)}$, where $A = \|B_\mu\|_{p\to q}$. The point here is to note that when $s = 0$, this inequality is, in fact, an equality (since the operator B_μ from $L^2(\mathbb{R}, \mathrm{d}g_\mu)$ to $\mathcal{B}^2_\mu$ is unitary). Then, by differentiating both sides of it at $s = 0^+$, we get a new inequality

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0^+} \|B_\mu f\|_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2,\mathrm{d}\nu_\mu)} \leqslant \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0^+} \left(A^s \|f\|_{L^{p_s}(\mathbb{R},\mathrm{d}g_\mu)}\right)$$

or

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0^+} \|B_\mu f\|_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2,\mathrm{d}\nu_\mu)} \leqslant (\log A)\|f\|_{L^2(\mathbb{R},\mathrm{d}g_\mu)} + \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0^+} f\|_{L^{p_s}(\mathbb{R},\mathrm{d}g_\mu)}. \tag{5.5}$$

Note that according to lemma 5.1, the hypotheses on f and on $B_\mu f$ guarantee the existence of the derivatives in this expression. Then we can use formula (5.2) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0^+} \|f\|_{L^{p_s}(\mathbb{R},\mathrm{d}g_\mu)} = (2^{-1} - p^{-1})\|f\|^{-1}_{L^2(\mathbb{R},\mathrm{d}g_\mu)} S_{L^2(\mathbb{R},\mathrm{d}g_\mu)}(f)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0^+} \|B_\mu f\|_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2,\mathrm{d}\nu_\mu)} = (2^{-1} - q^{-1})\|B_\mu f\|^{-1}_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2,\mathrm{d}\nu_\mu)} S_{L^2(\mathbb{C}\times\mathbb{Z}_2,\mathrm{d}\nu_\mu)}(B_\mu f).$$

Thus, inequality (5.5) becomes

$$(2^{-1} - q^{-1})\|B_\mu f\|^{-1}_{L^2(\mathbb{C}\times\mathbb{Z}_2,\mathrm{d}\nu_\mu)} S_{L^2(\mathbb{C}\times\mathbb{Z}_2,\mathrm{d}\nu_\mu)}(B_\mu f)$$
$$\leqslant (\log A)\|f\|_{L^2(\mathbb{R},\mathrm{d}g_\mu)} + (2^{-1} - p^{-1})\|f\|^{-1}_{L^2(\mathbb{R},\mathrm{d}g_\mu)} S_{L^2(\mathbb{R},\mathrm{d}g_\mu)}(f),$$

and finally, by using the fact $\|B_\mu f\|_{L^2(\mathbb{C}\times\mathbb{Z}_2,\mathrm{d}\nu_\mu)} = \|f\|_{L^2(\mathbb{R},\mathrm{d}g_\mu)}$, we obtain the inequality (5.4). □

Remark. This proof depends on the fact that B_μ is a unitary operator for $p = q = 2$ and $\lambda = 1$. We cannot extend this proof to the case $p = q = 2$ and $\lambda \neq 1$ by proposition 5.1

6. Logarithmic Sobolev inequalities

Throughout this section the parameter $\lambda \geqslant 1$ will be assumed.

The term 'Sobolev inequality' refers to an estimate of lower order derivatives of a function in terms of its higher order derivatives. Ever since the work of Sobolev ([Sob]), this kind of estimate has proven to be very useful in the theory of partial differential equations. (See [L-L], chapter 8.) An example of a Sobolev inequality for a function $f : \mathbb{R}^n \to \mathbb{C}$ is

$$S_n\|f\|^2_{L^q(\mathbb{R}^n,\mathrm{d}x)} \leqslant \|\mathrm{grad}\, f\|^2_{L^2(\mathbb{R}^n,\mathrm{d}x)},$$

where $n \geqslant 3$, $q = 2n(n-2)^{-1}$ and S_n a universal constant depending only on n. (See [L-L], p 156.)

In 1975, Gross ([G]) obtained the inequality

$$\int_{\mathbb{R}^n} |f(x)|^2 \log|f(x)|\, \mathrm{d}\nu(x) - \|f\|^2_{L^2(\mathbb{R}^n,\mathrm{d}\nu)} \log\|f\|_{L^2(\mathbb{R}^n,\mathrm{d}\nu)} \leqslant \int_{\mathbb{R}^n} |\mathrm{grad}\, f(x)|^2\, \mathrm{d}\nu(x),$$

valid for suitable functions $f : \mathbb{R}^n \to \mathbb{C}$, where $\mathrm{d}\nu$ is a Gaussian measure on $\mathbb{R}^n$. This inequality has the same flavour of the Sobolev inequality mentioned above, since on both right-hand sides appears the L^2 norm of grad f, and on the left-hand side appears an L^p norm of the function itself, with some mixed logs in the latter case. Gross refers to this result as a *logarithmic Sobolev inequality*, and this type of inequality has been shown since Gross' work in a variety of generalizations. In particular, in [Snt1] a logarithmic Sobolev inequality (LSI, for short) is obtained in the context of Segal–Bargmann analysis. Following the same sort of ideas, we will obtain in this section an LSI for the μ-deformed Segal–Bargmann space and its associated transform.

Recall that the two main steps in the development of the theory in the last section were first to have a Hausdorff–Young type inequality (in order to have an inequality between operator norms that are smooth functions of the corresponding Lebesgue indices), and second to use this inequality in order to obtain the Hirschman inequality (by applying the differentiation technique of Hirschman to the inequality of the first step). We will follow in this section the analogues of these steps by first proving another Hausdorff–Young type inequality and then using this inequality to obtain the LSI desired.

Instead of the Riesz–Thorin interpolation theorem we used to prove the Hausdorff–Young type inequality in the previous section, we will use here a generalization of it (Stein's theorem) which we quote next. Recall that a *simple function* is a measurable function f having a finite range $R \subset \mathbb{C}$ such that $f^{-1}(z)$ is a set of finite measure for every $z \in R$, $z \neq 0$.

Theorem 6.1 (Stein [St])**.** *Let* $(\Omega_j, \mathrm{d}\nu_j)$ *for* $j = 1, 2$ *be* σ*-finite measure spaces. Let* T *be a linear transformation which takes simple functions* $f : \Omega_1 \to \mathbb{C}$ *to measurable functions* $Tf : \Omega_2 \to \mathbb{C}$*. Let* $p_i, q_i \in [1, \infty], i = 0, 1$*. Then, for* $s \in [0, 1]$*, define* p_s *and* q_s *by* $p_s^{-1} = (1-s)p_0^{-1} + sp_1^{-1}$ *and* $q_s^{-1} = (1-s)q_0^{-1} + sq_1^{-1}$*. For* $i = 0, 1$*, suppose that* $u_i : \Omega_1 \to [0, \infty)$ *and* $k_i : \Omega_2 \to [0, \infty)$ *are measurable functions with the property that for all simple functions* $f : \Omega_1 \to \mathbb{C}$ *there exist finite non-negative constants* A_i *such that*

$$\|(Tf)k_i\|_{L^{q_i}(\Omega_2, \mathrm{d}\nu_2)} \leqslant A_i \|f u_i\|_{L^{p_i}(\Omega_1, \mathrm{d}\nu_1)}. \tag{6.1}$$

For $s \in [0, 1]$*, define functions* $u_s : \Omega_1 \to [0, \infty)$ *and* $k_s : \Omega_2 \to [0, \infty)$*, by* $u_s = u_0^{1-s}u_1^s$ *and* $k_s = k_0^{1-s}k_1^s$*. Then the transformation* T *can be extended uniquely to a linear transformation defined on the space of all* $f : \Omega_1 \to \mathbb{C}$ *that satisfy* $\|f u_s\|_{L^{p_s}(\Omega_1, \mathrm{d}\nu_1)} < \infty$ *in such a way that for all such* f *we have*

$$\|(Tf)k_s\|_{L^{q_s}(\Omega_2, \mathrm{d}\nu_2)} \leqslant A_0^{1-s} A_1^s \|f u_s\|_{L^{p_s}(\Omega_1, \mathrm{d}\nu_1)}. \tag{6.2}$$

We will need later the following result.

Lemma 6.1. *Let* $1 \leqslant q < 2\lambda$ *and* $0 \leqslant s \leqslant 1$*. Let the function* $\kappa_{\lambda,s} : \mathbb{C} \times \mathbb{Z}_2 \to [0, \infty)$ *be defined by*

$$\kappa_{\lambda,s}(z, 1) = \left(\frac{\lambda^{\frac{2\mu+3}{2}} K_{\mu-\frac{1}{2}}(\lambda|z|^2)}{K_{\mu-\frac{1}{2}}(|z|^2)} \right)^{sq^{-1}},$$

$$\kappa_{\lambda,s}(z, -1) = \left(\frac{\lambda^{\frac{2\mu+3}{2}} K_{\mu+\frac{1}{2}}(\lambda|z|^2)}{K_{\mu+\frac{1}{2}}(|z|^2)} \right)^{sq^{-1}}.$$

Then $\kappa_{\lambda,s} \in L^\infty(\mathbb{C} \times \mathbb{Z}_2)$*.*

Proof. We will prove that the restrictions of $\kappa_{\lambda,s}$ to each copy of $\mathbb{C}$ in $\mathbb{C} \times \mathbb{Z}_2$ are bounded functions in a neighbourhood of the origin and in a neighbourhood of infinity, from which the conclusion of the lemma follows. We begin by considering $\kappa_{\lambda,s}(z, 1)$ in a neighbourhood of $(0, 1)$. By applying (2.6) we find that if $0 \leqslant \mu < \frac{1}{2}$ we have that $\kappa_{\lambda,s}(z, 1) \cong \lambda^{\frac{(2\mu+1)s}{q}}$ as $|z| \to 0$, and if $\mu > \frac{1}{2}$ we have that $\kappa_{\lambda,s}(z, 1) \cong \lambda^{\frac{2s}{q}}$ as $|z| \to 0$. This shows that, for $\mu \neq \frac{1}{2}$, the function $\kappa_{\lambda,s}(z, 1)$ is bounded in a neighbourhood of $(0, 1)$. In the case $\mu = \frac{1}{2}$ we have by (2.7) that

$$\kappa_{\lambda,s}(z, 1) \cong \left(\frac{\lambda^2 \log \frac{2}{\lambda|z|^2}}{\log \frac{2}{|z|^2}} \right)^{sq^{-1}}.$$

But the right-hand side of this expression is bounded in a neighbourhood of the origin since it has the finite limit $\lambda^{\frac{2s}{q}}$ as $|z| \to 0$. Again using (2.6) we have that $\kappa_{\lambda,s}(z, -1) \cong \lambda^{\frac{s}{q}}$ as $|z| \to 0$ for all $\mu \geqslant 0$, which shows that $\kappa_{\lambda,s}(z, -1)$ is bounded near $(0, -1)$.

Finally, according to (2.8) we have that both $\kappa_{\lambda,s}(z,1)$ and $\kappa_{\lambda,s}(z,-1)$ are asymptotically equivalent as $|z| \to +\infty$ to

$$\left(\frac{\lambda^{\frac{2\mu+3}{2}} \left(\frac{\pi}{2\lambda}\right)^{\frac{1}{2}} |z|^{-1} \exp(-\lambda|z|^2)}{\left(\frac{\pi}{2}\right)^{\frac{1}{2}} |z|^{-1} \exp(-|z|^2)} \right)^{sq^{-1}} = \lambda^{\frac{(\mu+1)s}{q}} \exp\left(\frac{1-\lambda}{q} s|z|^2 \right),$$

which is a bounded function of z, since $\lambda \geqslant 1$. □

We will prove now a Hausdorff–Young type inequality as we did in theorem 5.1. Recall that in section 5 we worked with the operator B_μ from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^q(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_\mu)$ with p and q chosen in such a way that $\Pi_1 = (p^{-1}, q^{-1})$ is admissible. Then we used the Riesz–Thorin interpolation theorem to conclude that for all pairs $\left(p_s^{-1}, q_s^{-1}\right) = s\Pi_1 + (1-s)\Pi_2, 0 \leqslant s \leqslant 1$, in the line segment connecting $\Pi_2 = (2^{-1}, 2^{-1})$ and Π_1, the corresponding operator B_μ from $L^{p_s}(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^{q_s}(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_\mu)$ is bounded and that $1 \leqslant \|B_\mu\|_{p_s \to q_s} \leqslant \|B_\mu\|_{p\to q}^s$ for all $s \in [0, 1]$. Since the operator B_μ from $L^2(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^2(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_\mu)$ is isometric we have that Π_2 is admissible. Note that the measure $\mathrm{d}\nu_\mu$ of the spaces $L^{q_s}(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_\mu)$ is independent of the parameter $s \in [0, 1]$. What we will do now will be something like repeating this story in another setting, using the Stein's interpolation theorem instead of the Riesz–Thorin theorem, in such a way that we get the same sort of result: an inequality for the operator norm of the operators from the L^{p_s} spaces to the L^{q_s} spaces, such that when $s = 0$ this inequality becomes an equality. The price to be paid has to do with the measure of the L^{q_s} codomain spaces, which now will depend on the parameter s. The result is the following.

Theorem 6.2 (weighted Hausdorff–Young type inequality)**.** *Let p, q, λ be parameters as in (4.1). For $s \in [0, 1]$ let $\kappa_{\lambda,s} : \mathbb{C} \times \mathbb{Z}_2 \to [0, \infty)$ be the function defined in lemma 6.1, and p_s and q_s be defined by $p_s = ((1-s)2^{-1} + sp^{-1})^{-1}$ and $q_s = ((1-s)2^{-1} + sq^{-1})^{-1}$. Then for all $s \in [0, 1]$, the μ-deformed Segal–Bargmann transform B_μ is a bounded linear map from $L^{p_s}(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^{q_s}\left(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda}^s\right)$, where*

$$\mathrm{d}\nu_{\mu,\lambda}^s(z,1) = (\kappa_{\lambda,s}(z,1))^{q_s}\,\mathrm{d}\nu_{e,\mu}(z), \qquad \mathrm{d}\nu_{\mu,\lambda}^s(z,-1) = (\kappa_{\lambda,s}(z,-1))^{q_s}\,\mathrm{d}\nu_{o,\mu}(z).$$

Moreover, for $s \in [0, 1]$ and $f \in L^{p_s}(\mathbb{R}, \mathrm{d}g_\mu)$ we have that

$$\|B_\mu f\|_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda}^s)} \leqslant \|B_\mu\|_{p\to q}^s \|f\|_{L^{p_s}(\mathbb{R}, \mathrm{d}g_\mu)}. \tag{6.3}$$

Proof. First let us note that for $s = 0$ the measure $\mathrm{d}\nu_{\mu,\lambda}^s$ is simply $\mathrm{d}\nu_\mu$, while for $s = 1$ we have that

$$\begin{aligned}
\mathrm{d}\nu_{\mu,\lambda}^1(z,1) &= (\kappa_{\lambda,1}(z,1))^q\,\mathrm{d}\nu_{e,\mu}(z) \\
&= \frac{\lambda^{\frac{2\mu+3}{2}} K_{\mu-\frac{1}{2}}(\lambda|z|^2)}{K_{\mu-\frac{1}{2}}(|z|^2)} \frac{2^{\frac{1}{2}-\mu}}{\pi\Gamma\left(\mu+\frac{1}{2}\right)} K_{\mu-\frac{1}{2}}(|z|^2)|z|^{2\mu+1}\,\mathrm{d}x\,\mathrm{d}y \\
&= \lambda \frac{2^{\frac{1}{2}-\mu}}{\pi\Gamma\left(\mu+\frac{1}{2}\right)} K_{\mu-\frac{1}{2}}(\lambda|z|^2)\left|\lambda^{\frac{1}{2}} z\right|^{2\mu+1}\,\mathrm{d}x\,\mathrm{d}y \\
&= \mathrm{d}\nu_{\mu,\lambda}(z,1).
\end{aligned}$$

Similarly we have $\mathrm{d}\nu_{\mu,\lambda}^1(z,-1) = \mathrm{d}\nu_{\mu,\lambda}(z,-1)$. That is, the measure $\mathrm{d}\nu_{\mu,\lambda}^1$ is $\mathrm{d}\nu_{\mu,\lambda}$.

With the notation of Stein's theorem, we take $(\Omega_1, \mathrm{d}\nu_1) = (\mathbb{R}, \mathrm{d}g_\mu)$ and $(\Omega_2, \mathrm{d}\nu_2) = (\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_\mu)$. Take also $p_0 = q_0 = 2, p_1 = p, q_1 = q, u_0, u_1 : \mathbb{R} \to [0, \infty), u_0(t) = u_1(t) \equiv 1$, and $k_0 : \mathbb{C} \times \mathbb{Z}_2 \to [0, \infty), k_0(z, j) \equiv 1, j = -1, 1$. Define $k_1 : \mathbb{C} \times \mathbb{Z}_2 \to [0, \infty)$ as $k_1 := \kappa_{\lambda,1}$, where $\kappa_{\lambda,1}$ is defined in lemma 6.1.

For $s \in [0, 1]$, the function $u_s : \mathbb{R} \to [0, \infty)$ in Stein's theorem is $u_s = u_0^{1-s} u_1^s = 1$, and the function $k_s : \mathbb{C} \times \mathbb{Z}_2 \to [0, \infty)$ in Stein's theorem is $k_s = k_0^{1-s} k_1^s = k_1^s = \kappa_{\lambda,s}$, where $\kappa_{\lambda,s}$ is the function described in lemma 6.1.

Observe that

$$\|(B_\mu f)k_s\|_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu_\mu)} = \|B_\mu f\|_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu^s_{\mu,\lambda})}, \tag{6.4}$$

since

$$\begin{aligned}
\|(B_\mu f)k_s\|^{q_s}_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu_\mu)}
&= \int_{\mathbb{C}} |(B_{e,\mu} f)(z)|^{q_s} (\kappa_{\lambda,s}(z, 1))^{q_s} \, \mathrm{d}\nu_{e,\mu}(z) \\
&\quad + \int_{\mathbb{C}} |(B_{o,\mu} f)(z)|^{q_s} (\kappa_{\lambda,s}(z, -1))^{q_s} \, \mathrm{d}\nu_{o,\mu}(z) \\
&= \int_{\mathbb{C}} |(B_{e,\mu} f)(z)|^{q_s} \, \mathrm{d}\nu^s_{\mu,\lambda}(z, 1) + \int_{\mathbb{C}} |(B_{o,\mu} f)(z)|^{q_s} \, \mathrm{d}\nu^s_{\mu,\lambda}(z, -1) \\
&= \|B_\mu f\|^{q_s}_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu^s_{\mu,\lambda})}.
\end{aligned}$$

For $s = 0$, we have

$$\|(B_\mu f)k_0\|_{L^2(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu_\mu)} = \|B_\mu f\|_{L^2(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu_\mu)} = \|f\|_{L^2(\mathbb{R}, \mathrm{d}g_\mu)}.$$

Here the first equality is (6.4) and the second one is simply the fact that the μ-deformed Segal–Bargmann transform B_μ from $L^2(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^2(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_\mu)$ is an isometry. That is, the hypothesis (6.1) of Stein's theorem is satisfied for $i = 0$ with $A_0 = 1$.

For $s = 1$ we have

$$\|(B_\mu f)k_1\|_{L^q(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu_\mu)} = \|B_\mu f\|_{L^q(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})} \leqslant A_1 \|f\|_{L^p(\mathbb{R}, \mathrm{d}g_\mu)}.$$

Here the first equality is again (6.4) and the second one is justified by the fact that B_μ from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^q(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_{\mu,\lambda})$ is bounded, by theorem 4.1. Then, the hypothesis (6.1) of Stein's theorem is satisfied for $i = 1$ with $A_1 = \|B_\mu\|_{p\to q}$.

Thus, Stein's theorem allows us to conclude that the operator B_μ from $L^{p_s}(\mathbb{R}, \mathrm{d}g_\mu)$ to $L^{q_s}\left(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu^s_{\mu,\lambda}\right)$ is bounded and that

$$\|B_\mu f\|_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu^s_{\mu,\lambda})} \leqslant A_1^s \|f\|_{L^{p_s}(\mathbb{R}, \mathrm{d}g_\mu)},$$

as we wanted. □

The log-Sobolev inequality proved in [Snt1] involves the term $\langle f, Nf\rangle_{L^2(\mathbb{R}, \mathrm{d}g)}$, called the *Dirichlet energy* (in the space $L^2(\mathbb{R}, \mathrm{d}g)$), which is the quadratic form associated with the number (or energy) operator N. This operator is defined as $N = a^*a$, where a^* and a are the creation and annihilation operators, respectively, acting in the ground state representation $L^2(\mathbb{R}, \mathrm{d}g)$. The operators a^* and a can be defined by their action on the elements $\zeta_n(t) = 2^{-\frac{n}{2}}(n!)^{-\frac{1}{2}} H_n(t), n = 0, 1, 2, \ldots$ (where $H_n(t)$ is the nth Hermite polynomial), which form an orthonormal basis of $L^2(\mathbb{R}, \mathrm{d}g)$. The definitions are $a^*\zeta_n = (n + 1)^{\frac{1}{2}}\zeta_{n+1}$ and $a\zeta_n = n^{\frac{1}{2}}\zeta_{n-1}$, where $n = 0, 1, 2, \ldots$, and one defines $\zeta_{-1} = 0$. It turns out that a^* is the adjoint of a (with adequate definitions of their domains, which we do not give here). Observe that $N\zeta_n = n\zeta_n$, so ζ_n is an eigenvector of N and n is the corresponding eigenvalue. This justifies the name 'number operator' for N. Observe also that

$$\langle f, Nf\rangle_{L^2(\mathbb{R}, \mathrm{d}g)} = \langle af, af\rangle_{L^2(\mathbb{R}, \mathrm{d}g)} = \|af\|^2_{L^2(\mathbb{R}, \mathrm{d}g)} = \tfrac{1}{2}\|f'\|^2_{L^2(\mathbb{R}, \mathrm{d}g)}.$$

That is, the Dirichlet energy is, up to a constant, the norm (in the space $L^2(\mathbb{R}, \mathrm{d}g)$) of the derivative of the function f (belonging to the domain of N), which is the Dirichlet form of f.

Note that this is one of the ingredients of the Sobolev inequalities mentioned at the beginning of this section.

The Segal–Bargmann transform $B : L^2(\mathbb{R}, \mathrm{d}g) \to \mathcal{B}^2$ intertwines the action of a^* and a for the domain and codomain spaces in the sense that Ba^*B^{-1} and BaB^{-1} are the corresponding creation and annihilation operators in the Segal–Bargmann space $\mathcal{B}^2$. We will continue denoting these operators as a^* and a (acting on $\mathcal{B}^2$). It turns out that $(a^*f)(z) = zf(z)$, and $(af)(z) = f'(z)$, where f' is the complex derivative of the holomorphic function f. Observe that since B is unitary we have that $\langle f, Nf\rangle_{L^2(\mathbb{R},\mathrm{d}g)} = \langle Bf, \widetilde{N}Bf\rangle_{\mathcal{B}^2}$, where $\widetilde{N} = BNB^{-1}$ is the number operator in $\mathcal{B}^2$. That is, the Segal–Bargmann transform B preserves the Dirichlet energy (one says simply that 'B preserves energy').

For $\mu > -\frac{1}{2}$, the μ-deformed generalizations of the results above began to be considered in [Ros1], [Ros2] and [Marr], where the μ-deformed creation a_μ^* and annihilation a_μ operators in the μ-deformed quantum configuration space $L^2(\mathbb{R}, |t|^{2\mu}\,\mathrm{d}t)$ are defined. These definitions are given in terms of the μ-deformed position operator $(Q_\mu f)(t) = tf(t)$ and the μ-deformed momentum operator $(P_\mu f)(t) = -\mathrm{i}(D_\mu f)(t)$, where $(D_\mu f)(t) := f'(t) + \frac{\mu}{t}(f(t) - f(-t))$. We mention in passing that D_μ, which is a μ-deformation of the derivative, is a special case of a more general class of operators called Dunkl operators (see [Ros]). The definitions of a_μ^* and a_μ are $a_\mu^* = 2^{-\frac{1}{2}}(Q_\mu - \mathrm{i}P_\mu)$ and $a_\mu = 2^{-\frac{1}{2}}(Q_\mu + \mathrm{i}P_\mu)$. The corresponding μ-deformed creation and annihilation operators in $L^2(\mathbb{R}, \mathrm{d}g_\mu)$ can be defined by their action on the polynomials $\zeta_n^\mu(t) = 2^{-\frac{n}{2}}(n!)^{-1}(\gamma_\mu(n))^{\frac{1}{2}}H_n^\mu(t), n = 0, 1, 2, \ldots,$ (where $H_n^\mu(t)$ is the μ-deformed Hermite polynomial of degree n; see section 3), which form an orthonormal basis of $L^2(\mathbb{R}, \mathrm{d}g_\mu)$. The definitions are $a_\mu^*\zeta_n^\mu = (n + 1 + 2\mu\theta(n + 1))^{\frac{1}{2}}\zeta_{n+1}^\mu$ and $a_\mu\zeta_n^\mu = (n+2\mu\theta(n))^{\frac{1}{2}}\zeta_{n-1}^\mu$, where one defines $\zeta_{-1}^\mu = 0$. By considering the orthonormal basis $\{\xi_n^\mu\}_{n=0}^\infty$ of the μ-deformed Segal–Bargmann space $\mathcal{B}_\mu^2$, one can define the Segal–Bargmann transform $B_\mu : L^2(\mathbb{R}, \mathrm{d}g_\mu) \to \mathcal{B}_\mu^2$ as $B(\zeta_n^\mu) = \xi_n^\mu, n = 0, 1, 2, \ldots$. From this definition it is clear that B_μ is a unitary onto operator. It is easy to see that the creation and annihilation operators on $\mathcal{B}_\mu^2$ are $(a_\mu^* f)(z) = zf(z)$ and $a_\mu f = \widetilde{D}_\mu f$, respectively. Here $\widetilde{D}_\mu$ acts on holomorphic functions $f(z)$ as $\widetilde{D}_\mu f(z) := f'(z) + \frac{\mu}{z}(f(z) - f(-z))$, where f' is the complex derivative of f. The μ-deformed number operator on $\mathcal{B}_\mu^2$ is $\widetilde{N}_\mu = a_\mu^* a_\mu$, and one easily checks that $\widetilde{N}_\mu\xi_n^\mu = (n + 2\mu\theta(n))\xi_n^\mu, n = 0, 1, 2, \ldots$.

In [A-S.1] a μ-deformed energy E_μ is introduced for functions $f \in \mathcal{B}_\mu^2$, which appears as a term in a reverse log-Sobolev inequality proved there. The definition is

$$E_\mu(f) = E_{e,\mu}(f_e) + E_{o,\mu}(f_o), \tag{6.5}$$

where f_e and f_o are the even and odd parts of f, respectively, and

$$E_{e,\mu}(f_e) = \int_{\mathbb{C}} |f_e(z)|^2|z|^2\,\mathrm{d}\nu_{e,\mu}(z),$$
$$E_{o,\mu}(f_o) = \int_{\mathbb{C}} |f_o(z)|^2|z|^2\,\mathrm{d}\nu_{o,\mu}(z).$$

When $\mu = 0$ we have that $\mathrm{d}\nu_{e,0} = \mathrm{d}\nu_{o,0} = \mathrm{d}\nu_{\mathrm{Gauss}}$ and (6.5) becomes

$$E_0(f) = \int_{\mathbb{C}} |f_e(z)|^2|z|^2\,\mathrm{d}\nu_{\mathrm{Gauss}}(z) + \int_{\mathbb{C}} |f_o(z)|^2|z|^2\,\mathrm{d}\nu_{\mathrm{Gauss}}(z)$$
$$= \int_{\mathbb{C}} |f(z)|^2|z|^2\,\mathrm{d}\nu_{\mathrm{Gauss}}(z).$$

(In the last equality we used that $zf_e(z) \in \mathcal{B}_o^2, zf_o(z) \in \mathcal{B}_e^2$, and that $\mathcal{B}_o^2$ and $\mathcal{B}_e^2$ are orthogonal subspaces of $\mathcal{B}^2$.)

In [Bar] it is proved that

$$\int_{\mathbb{C}} |f(z)|^2 |z|^2 \, d\nu_{\mathrm{Gauss}}(z) = \|f\|^2_{\mathcal{B}^2} + \langle f, \widetilde{N} f\rangle_{\mathcal{B}^2}, \tag{6.6}$$

where $f \in \mathcal{B}^2$. This result (Bargmann identity) shows that in the case $\mu = 0$ the μ-deformed energy defined above is related with the Dirichlet energy $\langle f, \widetilde{N} f\rangle_{\mathcal{B}^2}$ for $f \in \mathcal{B}^2$. The μ-deformed number operator N_μ acting in $\mathcal{B}^2_\mu$ and its corresponding quadratic form $\langle f, N_\mu f\rangle_{\mathcal{B}^2_\mu}$, which can be identified as a μ-deformed Dirichlet form, seem to have been introduced in [A-S.1]. The relation of this Dirichlet energy and the μ-deformed energy $E_\mu(f)$ is studied in [A-S.2].

In the log-Sobolev inequality we will prove in this section there appears a new mathematical object that it is natural to relate with the energy. We will call it *dilation energy*, and its definition is the following.

Definition 6.1. *The dilation energy of an even function $f \in \mathcal{B}^2_{e,\mu}$ is defined by*

$$E_{e,\mu,\lambda}(f) = \int_{\mathbb{C}} |f(z)|^2 \log\left(\frac{K_{\mu-\frac{1}{2}}(|z|^2)}{K_{\mu-\frac{1}{2}}(\lambda|z|^2)}\right) d\nu_{e,\mu}(z).$$

The dilation energy of an odd function $f \in \mathcal{B}^2_{o,\mu}$ is defined by

$$E_{o,\mu,\lambda}(f) = \int_{\mathbb{C}} |f(z)|^2 \log\left(\frac{K_{\mu+\frac{1}{2}}(|z|^2)}{K_{\mu+\frac{1}{2}}(\lambda|z|^2)}\right) d\nu_{o,\mu}(z).$$

The dilation energy of a function $f \in \mathcal{B}^2_\mu$ is defined by

$$E_{\mu,\lambda}(f) = E_{e,\mu,\lambda}(f_e) + E_{o,\mu,\lambda}(f_o). \tag{6.7}$$

Observe that the fact that $\lambda \geqslant 1$ and the decreasing property of $K_\alpha(x)$ for $x \in \mathbb{R}^+$ imply that $\log \frac{K_\alpha(|z|^2)}{K_\alpha(\lambda|z|^2)} \geqslant 0$, so we have that $E_{\mu,\lambda}(f) \geqslant 0$.

When $\mu = 0$ we can use (2.5) to obtain

$$\begin{aligned} E_{0,\lambda}(f) &= \int_{\mathbb{C}} \left(|f_e(z)|^2 \log\left(\frac{K_{-\frac{1}{2}}(|z|^2)}{K_{-\frac{1}{2}}(\lambda|z|^2)}\right) + |f_o(z)|^2 \log\left(\frac{K_{\frac{1}{2}}(|z|^2)}{K_{\frac{1}{2}}(\lambda|z|^2)}\right)\right) d\nu_{\mathrm{Gauss}}(z) \\ &= \int_{\mathbb{C}} (|f_e(z)|^2 + |f_o(z)|^2) \log\left(\frac{\left(\frac{\pi}{2|z|^2}\right)^{\frac{1}{2}} \exp(-|z|^2)}{\left(\frac{\pi}{2\lambda|z|^2}\right)^{\frac{1}{2}} \exp(-\lambda|z|^2)}\right) d\nu_{\mathrm{Gauss}}(z) \\ &= \int_{\mathbb{C}} (|f_e(z)|^2 + |f_o(z)|^2)\big(\log \lambda^{\frac{1}{2}} + (\lambda - 1)|z|^2\big) d\nu_{\mathrm{Gauss}}(z) \\ &= \big(\log \lambda^{\frac{1}{2}}\big)\|f\|^2_{\mathcal{B}^2} + (\lambda - 1)\left(\int_{\mathbb{C}} |f(z)|^2 |z|^2 \, d\nu_{\mathrm{Gauss}}(z)\right). \end{aligned}$$

By using the Bargmann identity (6.6), we can write

$$E_{0,\lambda}(f) = \big(\log \lambda^{\frac{1}{2}} + \lambda - 1\big)\|f\|^2_{\mathcal{B}^2} + (\lambda - 1)\langle f, \widetilde{N} f\rangle_{\mathcal{B}^2},$$

which shows that, in the case $\mu = 0$, the dilation energy $E_{\mu,\lambda}(f)$ is related with the Dirichlet energy $\langle f, \widetilde{N} f\rangle_{\mathcal{B}^2}$, where $f \in \mathcal{B}^2$.

In fact, for any $\mu > 0$, the dilation energy $E_{\mu,\lambda}$ is related with the μ-deformed energy E_μ, as we will see now.

We will use the following formula for $K_\nu(x)$, valid for $x \in \mathbb{R}^+$ and $\nu > -\frac{1}{2}$ (see [Wat], p 207.):

$$K_\nu(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \left(\sum_{k=0}^{n-1} \frac{\Gamma\left(\nu + \frac{1}{2} + k\right)}{k!\Gamma\left(\nu + \frac{1}{2} - k\right)(2x)^k} + \frac{\eta(x)\Gamma\left(\nu + \frac{1}{2} + n\right)}{n!\Gamma\left(\nu + \frac{1}{2} - n\right)(2x)^n} \right). \tag{6.8}$$

Here $\eta(x)$ is a function of x, $0 \leqslant \eta(x) \leqslant 1$, and the non-negative integer n is chosen such that $n - 1 < \nu - \frac{1}{2} \leqslant n$.

From (6.8) we obtain that

$$\frac{K_{\mu-\frac{1}{2}}(|z|^2)}{K_{\mu-\frac{1}{2}}(\lambda|z|^2)} = \lambda^{\frac{1}{2}} \exp((\lambda - 1)|z|^2) S(z, \mu, \lambda),$$

where for $z \neq 0$

$$S(z, \mu, \lambda) = \frac{\sum_{k=0}^{m-1} \frac{\Gamma(\mu+k)}{k!\Gamma(\mu-k)(2|z|^2)^k} + \eta(|z|^2) \frac{\Gamma(\mu+m)}{m!\Gamma(\mu-m)(2|z|^2)^m}}{\sum_{k=0}^{m-1} \frac{\Gamma(\mu+k)}{k!\Gamma(\mu-k)(2\lambda|z|^2)^k} + \eta(\lambda|z|^2) \frac{\Gamma(\mu+m)}{m!\Gamma(\mu-m)(2\lambda|z|^2)^m}},$$

$\eta(|z|^2), \eta(\lambda|z|^2) \in [0, 1]$, and $m \in \mathbb{N} \cup \{0\}$ is such that $m < \mu \leqslant m + 1$.

Similarly we have that

$$\frac{K_{\mu+\frac{1}{2}}(|z|^2)}{K_{\mu+\frac{1}{2}}(\lambda|z|^2)} = \lambda^{\frac{1}{2}} \exp((\lambda - 1)|z|^2) T(z, \mu, \lambda),$$

where for $z \neq 0$

$$T(z, \mu, \lambda) = \frac{\sum_{k=0}^{n-1} \frac{\Gamma(\mu+1+k)}{k!\Gamma(\mu+1-k)(2|z|^2)^k} + \eta(|z|^2) \frac{\Gamma(\mu+1+n)}{n!\Gamma(\mu+1-n)(2|z|^2)^n}}{\sum_{k=0}^{n-1} \frac{\Gamma(\mu+1+k)}{k!\Gamma(\mu+1-k)(2\lambda|z|^2)^k} + \eta(\lambda|z|^2) \frac{\Gamma(\mu+1+n)}{n!\Gamma(\mu+1-n)(2\lambda|z|^2)^n}},$$

$\eta(|z|^2), \eta(\lambda|z|^2) \in [0, 1]$, and $n \in \mathbb{N} \cup \{0\}$ is such that $n - 1 < \mu \leqslant n$.

Thus, the dilation energy (6.7) can be written as

$$\begin{aligned} E_{\mu,\lambda}(f) &= \int_{\mathbb{C}} |f_e(z)|^2 \log\left(\lambda^{\frac{1}{2}} \exp((\lambda - 1)|z|^2) S(z, \mu, \lambda)\right) \mathrm{d}\nu_{e,\mu}(z) \\ &\quad + \int_{\mathbb{C}} |f_o(z)|^2 \log\left(\lambda^{\frac{1}{2}} \exp((\lambda - 1)|z|^2) T(z, \mu, \lambda)\right) \mathrm{d}\nu_{o,\mu}(z) \\ &= \left(\log \lambda^{\frac{1}{2}}\right) \|f\|^2_{\mathcal{B}^2_\mu} + (\lambda - 1) \left(\int_{\mathbb{C}} |f_e(z)|^2 |z|^2 \, \mathrm{d}\nu_{e,\mu}(z) + \int_{\mathbb{C}} |f_o(z)|^2 |z|^2 \, \mathrm{d}\nu_{o,\mu}(z) \right) \\ &\quad + \int_{\mathbb{C}} |f_e(z)|^2 (\log S(z, \mu, \lambda)) \, \mathrm{d}\nu_{e,\mu}(z) + \int_{\mathbb{C}} |f_o(z)|^2 (\log T(z, \mu, \lambda)) \, \mathrm{d}\nu_{o,\mu}(z). \end{aligned}$$

That is, for any $\mu \geqslant 0$ and $\lambda \geqslant 1$, we have that the dilation energy $E_{\mu,\lambda}(f)$ of a function $f \in \mathcal{B}^2_\mu$ is related with the μ-deformed energy $E_\mu(f)$ by

$$E_{\mu,\lambda}(f) = \left(\log \lambda^{\frac{1}{2}}\right) \|f\|^2_{\mathcal{B}^2_\mu} + (\lambda - 1) E_\mu(f) + \rho(\mu, \lambda, f),$$

where

$$\rho(\mu, \lambda, f) = \int_{\mathbb{C}} |f_e(z)|^2 (\log S(z, \mu, \lambda)) \, \mathrm{d}\nu_{e,\mu}(z) + \int_{\mathbb{C}} |f_o(z)|^2 (\log T(z, \mu, \lambda)) \, \mathrm{d}\nu_{o,\mu}(z).$$

By examining the last factor in (6.8) we see that $S(z, \mu, \lambda) \to 1$ as $|z| \to \infty$. This also follows from (2.8). Similarly, $T(z, \mu, \lambda) \to 1$ as $|z| \to \infty$. Moreover, $S(z, \mu, \lambda) \neq 0$ for all $z \neq 0$, since otherwise $K_{\mu-1/2}(|z|^2) = 0$, which is known to be false. Similarly, $T(z, \mu, \lambda) \neq 0$ for all $z \neq 0$. It is then not hard to see that there exist constants $0 < A_{\mu,\lambda} < B_{\mu,\lambda}$ such that

$$A_{\mu,\lambda} \|f\|^2_{\mathcal{B}^2_\mu} \leqslant \rho(\mu, \lambda, f) \leqslant B_{\mu,\lambda} \|f\|^2_{\mathcal{B}^2_\mu}$$

for all $f \in \mathcal{B}^2_\mu$. It follows for $\lambda > 1$ that the quadratic forms $E_{\mu,\lambda}(f)$ and $E_\mu(f)$ in $f \in \mathcal{B}^2_\mu$ are equivalent, modulo terms that are multiples of $\|f\|^2_{\mathcal{B}^2_\mu}$. Of course, for $\lambda = 1$ we have $E_{\mu,\lambda}(f) = 0$ for all $f \in \mathcal{B}^2_\mu$.

Now we go to the main result of this section.

Theorem 6.3 (Logarithmic Sobolev Inequalities)**.** *Let p, q be such that*

$$1 \leqslant q < 2\lambda \qquad \text{and} \qquad p > 1 + \frac{q}{2\lambda}.$$

Let $f \in L^{2+\zeta}(\mathbb{R}, \mathrm{d}g_\mu)$, where $\zeta > 0$, be such that $B_\mu f \in L^{2+\xi}(\mathbb{C} \times Z_2, \mathrm{d}\nu_\mu)$, where $\xi > 0$. Then we have the logarithmic Sobolev inequality

$$(2^{-1} - q^{-1}) S_{L^2(\mathbb{C}\times Z_2, \mathrm{d}\nu_\mu)}(B_\mu f) - (2^{-1} - p^{-1}) S_{L^2(\mathbb{R}, \mathrm{d}g_\mu)}(f) \leqslant \frac{1}{q} E_{\mu,\lambda}(B_\mu f) + \left(\log \|B_\mu\|_{p\to q} - \frac{2\mu + 3}{2q} \log \lambda \right) \|f\|^2_{L^2(\mathbb{R}, \mathrm{d}g_\mu)}. \tag{6.9}$$

Remark. Consider the subspace S of $L^2(\mathbb{R}, \mathrm{d}g_\mu)$ consisting of all f such that $f \in L^{2+\zeta}(\mathbb{R}, \mathrm{d}g_\mu)$ for some $\zeta > 0$ and $B_\mu f \in L^{2+\xi}(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_\mu)$ for some $\xi > 0$. Then S is dense in $L^2(\mathbb{R}, \mathrm{d}g_\mu)$. To see this, first observe that the polynomials ζ^μ_n are in $L^{2+\alpha}(\mathbb{R}, \mathrm{d}g_\mu)$ for every $\alpha > 0$, since the density of the measure contains a Gaussian factor which dominates the integrand near infinity. Now $B_\mu \zeta^\mu_n = \xi^\mu_n$ as we already know. But ξ^μ_n is a monomial and so $\xi^\mu_n \in L^{2+\beta}(\mathbb{C} \times \mathbb{Z}_2, \mathrm{d}\nu_\mu)$ for every $\beta > 0$, since again the measure goes to zero fast enough to guarantee convergence of the integral. Therefore, $\zeta^\mu_n \in S$ for every integer $n \geqslant 0$. But the set of finite linear combinations of the ζ^μ_n forms a subspace of S which itself is dense in $L^2(\mathbb{R}, \mathrm{d}g_\mu)$, since the ζ^μ_n are an orthonormal basis of $L^2(\mathbb{R}, \mathrm{d}g_\mu)$. And this shows that S is dense. Consequently, theorems 5.2 and 6.3 hold for functions in a dense subspace, namely S, of $L^2(\mathbb{R}, \mathrm{d}g_\mu)$. We fully expect that the results of these two theorems hold for all functions in $L^2(\mathbb{R}, \mathrm{d}g_\mu)$.

Proof of theorem 6.3. We will use the inequality (6.3), which combined with (6.4) tells us that for $s \in [0, 1]$ we have that

$$\|(B_\mu f) k_s\|_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu_\mu)} \leqslant A^s \|f\|_{L^{p_s}(\mathbb{R}, \mathrm{d}g_\mu)}, \tag{6.10}$$

where $A = \|B_\mu\|_{p\to q}$ and f is as in the hypotheses of the theorem. Also p_s and q_s are as in theorem 6.1. Observe that when $s = 0$, (6.10) becomes an equality, so we can differentiate (6.10) at $s = 0^+$ to obtain the new inequality

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0^+} \|(B_\mu f) k_s\|_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu_\mu)} \leqslant \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0^+} (A^s \|f\|_{L^{p_s}(\mathbb{R}, \mathrm{d}g_\mu)})$$

or

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0^+} \|(B_\mu f) k_s\|_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2, \mathrm{d}\nu_\mu)} \leqslant (\log A) \|f\|_{L^2(\mathbb{R}, \mathrm{d}g_\mu)} + \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0^+} \|f\|_{L^{p_s}(\mathbb{R}, \mathrm{d}g_\mu)}. \tag{6.11}$$

The hypothesis on f allows us to use lemma 5.1 and obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0^+} \|f\|_{L^{p_s}(\mathbb{R}, \mathrm{d}g_\mu)} = (2^{-1} - p^{-1}) \|f\|^{-1}_{L^2(\mathbb{R}, \mathrm{d}g_\mu)} S_{L^2(\mathbb{R}, \mathrm{d}g_\mu)}(f).$$

The hypothesis on $B_\mu f$ and lemma 6.1 imply $(B_\mu f) k_s \in L^{2+\xi}(\mathbb{C} \times Z_2, \mathrm{d}\nu_\mu)$ for $s \in [0, 1]$, and so we can also use lemma 5.1 to obtain the derivative of the left-hand side of (6.11). Observe that in this case the function F of lemma 5.1 is not a constant function. Thus, in this case formula (5.3) gives us

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0^+} \|(B_\mu f)k_s\|_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2,\mathrm{d}\nu_\mu)} = \|B_\mu f\|^{-1}_{L^2(\mathbb{C}\times Z_2,\mathrm{d}\nu_\mu)}(2^{-1}-q^{-1})S_{L^2(\mathbb{C}\times Z_2,\mathrm{d}\nu_\mu)}(B_\mu f) + \|B_\mu f\|^{-1}_{L^2(\mathbb{C}\times Z_2,\mathrm{d}\nu_\mu)}\operatorname{Re}\langle F'(0),(\operatorname{sgn} B_\mu f)|B_\mu f|\rangle,$$

where the derivative $F'(0)$ is

$$F'(0) = (B_\mu f)\,\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0^+} k_1^s = (B_\mu f)\log k_1,$$

and so

$$\begin{aligned}\operatorname{Re}\langle F'(0),(\operatorname{sgn} B_\mu f)|B_\mu f|\rangle &= \operatorname{Re}\int_{\mathbb{C}\times\mathbb{Z}_2} F'(0)\overline{(\operatorname{sgn}(B_\mu f))}|B_\mu f|\mathrm{d}\nu_\mu\\ &= \operatorname{Re}\int_{\mathbb{C}\times\mathbb{Z}_2}(\log k_1)(B_\mu f)\overline{B_\mu f}\,\mathrm{d}\nu_\mu\\ &= \int_{\mathbb{C}\times\mathbb{Z}_2}(\log k_1)|B_\mu f|^2\,\mathrm{d}\nu_\mu.\end{aligned}$$

Explicitly we have that

$$\begin{aligned}\operatorname{Re}\langle F'(0),(\operatorname{sgn} B_\mu f)|B_\mu f|\rangle &= \int_{\mathbb{C}}\log\left(\frac{\lambda^{\frac{2\mu+3}{2}}K_{\mu-\frac12}(\lambda|z|^2)}{K_{\mu-\frac12}(|z|^2)}\right)^{q^{-1}}|(B_{e,\mu}f)(z)|^2\,\mathrm{d}\nu_{e,\mu}(z)\\ &\quad+\int_{\mathbb{C}}\log\left(\frac{\lambda^{\frac{2\mu+3}{2}}K_{\mu+\frac12}(\lambda|z|^2)}{K_{\mu+\frac12}(|z|^2)}\right)^{q^{-1}}|(B_{o,\mu}f)(z)|^2\,\mathrm{d}\nu_{o,\mu}(z)\\ &= \frac{2\mu+3}{2q}(\log\lambda)\|B_\mu f\|^2_{L^2(\mathbb{C}\times Z_2,\mathrm{d}\nu_\mu)}-\frac{1}{q}E_{\mu,\lambda}(B_\mu f).\end{aligned}$$

Thus we have that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0^+}\|(B_\mu f)k_s\|_{L^{q_s}(\mathbb{C}\times\mathbb{Z}_2,\mathrm{d}\nu_\mu)}\\ &\quad=\|B_\mu f\|^{-1}_{L^2(\mathbb{C}\times Z_2,\mathrm{d}\nu_\mu)}\begin{pmatrix}(2^{-1}-q^{-1})S_{L^2(\mathbb{C}\times Z_2,\mathrm{d}\nu_\mu)}(B_\mu f)\\ +\frac{2\mu+3}{2q}(\log\lambda)\|B_\mu f\|^2_{L^2(\mathbb{C}\times Z_2,\mathrm{d}\nu_\mu)}-\frac1q E_{\mu,\lambda}(B_\mu f)\end{pmatrix}.\end{aligned}$$

So the inequality (6.11) becomes

$$\begin{aligned}&\|B_\mu f\|^{-1}_{L^2(\mathbb{C}\times Z_2,\mathrm{d}\nu_\mu)}\begin{pmatrix}(2^{-1}-q^{-1})S_{L^2(\mathbb{C}\times Z_2,\mathrm{d}\nu_\mu)}(B_\mu f)\\ +\frac{2\mu+3}{2q}(\log\lambda)\|B_\mu f\|^2_{L^2(\mathbb{C}\times Z_2,\mathrm{d}\nu_\mu)}-\frac1q E_{\mu,\lambda}(B_\mu f)\end{pmatrix}\\ &\quad\leqslant(\log A)\|f\|_{L^2(\mathbb{R},\mathrm{d}g_\mu)}+(2^{-1}-p^{-1})\|f\|^{-1}_{L^2(\mathbb{R},\mathrm{d}g_\mu)}S_{L^2(\mathbb{R},\mathrm{d}g_\mu)}(f),\end{aligned}$$

and finally, by using that $\|B_\mu f\|_{L^2(\mathbb{C}\times Z_2,\mathrm{d}\nu_\mu)} = \|f\|_{L^2(\mathbb{R},\mathrm{d}g_\mu)}$, we get

$$\begin{aligned}&(2^{-1}-q^{-1})S_{L^2(\mathbb{C}\times Z_2,\mathrm{d}\nu_\mu)}(B_\mu f)-(2^{-1}-p^{-1})S_{L^2(\mathbb{R},\mathrm{d}g_\mu)}(f)\leqslant\frac1q E_{\mu,\lambda}(B_\mu f)\\ &\quad+\left(\log\|B_\mu\|_{p\to q}-\frac{2\mu+3}{2q}\log\lambda\right)\|f\|^2_{L^2(\mathbb{R},\mathrm{d}g_\mu)},\end{aligned}$$

which is (6.9). □

Observe that, in the limiting case $\lambda = 1$, we have that $E_{\mu,\lambda}(B_\mu f) = 0$, and then the log-Sobolev inequality (6.9) becomes

$$\left(\frac{1}{2} - \frac{1}{q}\right) S_{L^2(\mathbb{C}\times Z_2, \mathrm{d}\nu_\mu)}(B_\mu f) \leqslant \left(\frac{1}{2} - \frac{1}{p}\right) S_{L^2(\mathbb{R},\mathrm{d}g_\mu)}(f) + (\log\|B_\mu\|_{p\to q})\|f\|^2_{L^2(\mathbb{R},\mathrm{d}g_\mu)},$$

which is the Hirschman inequality (5.4) we proved in the previous section.

In the case $\mu = 0$ the inequality (6.9) becomes

$$(2^{-1} - q^{-1})S_{L^2(\mathbb{C},\mathrm{d}\nu_{\mathrm{Gauss}})}(Bf) - (2^{-1} - p^{-1})S_{L^2(\mathbb{R},\mathrm{d}g)}(f) \leqslant \frac{1}{q}E_{0,\lambda}(Bf) + \left(\log\|B\|_{p\to q} - \frac{3}{2q}\log\lambda\right)\|f\|^2_{L^2(\mathbb{R},\mathrm{d}g)}$$

or

$$(2^{-1} - q^{-1})S_{L^2(\mathbb{C},\mathrm{d}\nu_{\mathrm{Gauss}})}(Bf) - (2^{-1} - p^{-1})S_{L^2(\mathbb{R},\mathrm{d}g)}(f) \leqslant \frac{1}{q}\left(\left(\log\lambda^{\frac{1}{2}} + \lambda - 1\right)\|Bf\|^2_{\mathcal{B}^2} + (\lambda - 1)\langle Bf, \widetilde{N}Bf\rangle_{\mathcal{B}^2}\right) + \left(\log\|B\|_{p\to q} - \frac{3}{2q}\log\lambda\right)\|f\|^2_{L^2(\mathbb{R},\mathrm{d}g)}.$$

By using that $\|Bf\|_{\mathcal{B}^2} = \|f\|^2_{L^2(\mathbb{R},\mathrm{d}g)}$ and $\langle Bf, \widetilde{N}Bf\rangle_{\mathcal{B}^2} = \langle f, Nf\rangle_{L^2(\mathbb{R},\mathrm{d}g)}$, we can write the last expression as

$$(2^{-1} - q^{-1})S_{L^2(\mathbb{C},\mathrm{d}\nu_{\mathrm{Gauss}})}(Bf) - (2^{-1} - p^{-1})S_{L^2(\mathbb{R},\mathrm{d}g)}(f) \leqslant \left(-\frac{1}{q}\log\lambda + \frac{\lambda - 1}{q} + \log\|B\|_{p\to q}\right)\|f\|^2_{L^2(\mathbb{R},\mathrm{d}g)} + \frac{\lambda - 1}{q}\langle f, Nf\rangle_{L^2(\mathbb{R},\mathrm{d}g)},$$

which is the log-Sobolev inequality in [Snt1], up to some identifications in the coefficients of the terms of the right-hand side (for example, the weight a that appears in [Snt1] can be identified with $\lambda - 1$).

7. Concluding remarks

In this section we present some of the lines along which this work can be continued.

(1) The μ-deformed theory presented in [Ros1], [Ros2] and [Marr] is valid for $\mu > -\frac{1}{2}$. Nevertheless, the inequality (2.2) was proved only for non-negative values of μ, and this inequality is fundamental in the proof of the theorem 4.1, and then in the proofs of results of the remaining sections. We leave as open questions if these results (sections 4, 5 and 6) are also valid for $-\frac{1}{2} < \mu < 0$.

(2) Theorem 4.1 establishes that if $p \in (1, \infty]$, $q \in [1, \infty)$, and $\lambda > \frac{1}{2}$ are such that the inequalities $p > 1 + \frac{q}{2\lambda}$ and $1 \leqslant q < 2\lambda$ hold, then the μ-deformed Segal–Bargmann transform B_μ is a bounded operator from $L^p(\mathbb{R}, \mathrm{d}g_\mu)$ to $\mathcal{B}^q_{\mu,\lambda}$. For p, q, and λ not satisfying the above mentioned inequalities we know little about the boundedness of B_μ. We suspect that if either of the inequalities $q > 2\lambda$ or $p < 1 + \frac{q}{2\lambda}$ holds, then B_μ is not bounded (for the corresponding values of p, q and λ), since this is the case when $\mu = 0$ and $\lambda = 1$ (see corollary 7.2 in [Snt1]), but in the general situation we consider in this work this remains as an open question.

Acknowledgments

The first author wishes to thank firstly CIMAT and the Mittag-Leffler Institute (Djursholm, Sweden) for supporting his attendance in the Fall Program *Partial Differential Equations and Spectral Theory* at the Mittag-Leffler Institute (October-December 2002) and secondly the Universidad Panamericana (Mexico City) for giving him the support for being a full-time doctoral student at CIMAT from 2001 to 2004. The second author thanks Larry Thomas for many useful comments over the course of the years. Both authors thank Shirley Bromberg, Pavel Naumkin, Roberto Quezada Batalla and Carlos Villegas-Blas for valuable comments. The research of CJPRV was partially supported by grant 165628, CONACyT (Mexico) and that of SBS was partially supported by grants 32146-E and P-42227-F, CONACyT (Mexico).

References

[A-S.1] Angulo C and Sontz S B 2006 Reverse inequalities in μ-deformed Segal–Bargmann analysis *J. Math. Phys.* **47** 042103

[A-S.2] Angulo C and Sontz S B 2005 *Direct and Reverse Log-Sobolev Inequalities in μ-deformed Segal–Bargmann Analysis* (*Preprint*)

[Bar] Bargmann V 1961 On a Hilbert space of analytic functions and an associated integral transform: I *Commun. Pure Appl. Math.* **14** 187–214

[Be] Beckner W 1975 Inequalities in Fourier analysis *Ann. Math.* **102** 159–82

[B-O] Ben Saïd S and Ørsted B 2006 Segal–Bargmann transforms associated with Coxeter groups *Mathematische Annalen* **334** (2) 281–323 (*Preprint* http://bib.mathematics.dk/preprint.php?lang=en&id=IMADA-PP-2004-12, p 44)

[B-S] Bennett C and Sharpley R 1988 *Interpolation of Operators* (New York: Academic)

[Fol] Folland G B 1989 *Harmonic Analysis in Phase Space (Annals of Mathematics Studies* vol 122*)* (Princeton, NJ: Princeton University Press)

[G] Gross L 1975 Logarithmic Sobolev inequalities *Am. J. Math.* **97** 1061–83

[Hall] Hall B 2000 Holomorphic methods in analysis and mathematical physics *First Summer School in Analysis and Mathematical Physics* (*Contemp. Math.* vol 260*)* ed S Pérez-Esteva and C Villegas-Blas (Providence, RI: American Mathematical Society) pp 1–59

[Hir] Hirschman I I Jr 1957 A note on entropy *Am. J. Math.* **79** 152–6

[J] Jörgens K 1982 *Linear Integral Operators* (*Advanced Publishing Program*) (Boston, MA: Pitman)

[Leb] Lebedev N N 1972 *Special Functions and Their Applications* (New York: Dover)

[L-L] Lieb E H and Loss M 1997 Analysis (*Graduate Studies in Mathematics* vol 14*)* (Providence, RI: American Mathematical Society)

[Marr] Marron C S 1994 Semigroups and the Bose-like oscillator *PhD Dissertation* presented to the Graduate Faculty of the University of Virginia

[Pi] Pita C 2004 unpublished notes

[P-S] Pita C and Sontz S B 2006 On Shannon entropies in μ-deformed Segal–Bargmann Analysis *J. Math. Phys.* **47** 032101

[R-S] Reed M and Simon B 1980 *Methods of Modern Mathematical Physics, Volume I: Functional Analysis* (New York: Academic) (revised and enlarged edn)

[Ros1] Rosenblum M 1994 Generalized Hermite polynomials and the Bose-like oscillator calculus *Nonselfadjoint Operators and Related Topics* ed A Feintuch and I Gohberg (Basel: Birkhäuser)

[Ros2] Rosenblum M 1994 Operators associated with the Bose-like oscillator, unpublished manuscript

[Ros] Rösler M 2003 *Dunkl Operators: Theory and Applications* (*Lecture Notes in Mathematics* vol 1817*)* ed E Koelink and W Van Assche (Berlin: Springer) pp 93–135

[Seg1] Segal I E 1963 *Mathematical Problems of Relativistic Physics* (Providence, RI: American Mathematical Society)

[Seg2] Segal I E 1978 The complex-wave representation of the free Boson field *Topics in Functional Analysis, Essays Dedicated to M.G. Krein on the Occasion of His 70th Birthday* ed I Gohberg and M Kac (New York: Academic)

[Sha] Shannon C and Weaver W 1949 *A Mathematical Theory of Communication* (Urbana: University of Illinois Press)

[Sob] Sobolev S L 1938 On a theorem of functional analysis *Math. Sb., NS* **4** 471–9
Sobolev S L 1963 *AMS Transl. (Ser. 2)* **34** 39–68 (Engl. Transl.)

[Snt1] Sontz S B 1998 Entropy and the Segal–Bargmann transform *J. Math. Phys.* **39** 2402–17

[Snt2] Sontz S B 1999 A reverse log–Sobolev inequality in the Segal-Bargmann space *J. Math. Phys.* **40** 1677–95

[Snt3] Sontz S B 2005 *On μ-deformed Segal–Bargmann Analysis and its Heat Kernel* (*Preprint*)

[St] Stein E 1956 Interpolation of linear operators *Trans. Am. Math. Soc.* **83** 482–92

[Wat] Watson G N 1944 *A Treatise on the Theory of Bessel Functions* 2nd edn (Cambridge: Cambridge University Press)

[We] Weiss G 1965 Harmonic analysis *Studies in Real and Complex Analysis (MAA Studies in Mathematics* vol 3*)* ed I I Hirschman Jr (Washington, DC: The Mathematical Association of America)

[Wig] Wigner E P 1950 Do the equations of motion determine the quantum mechanical commutation relations? *Phys. Rev.* **77** 711–2